Exponential Brownian Motion & Approximation Theory

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November 19, 2019

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Consider exponential Brownian motion

$$S(t) = e^{(r-\sigma^2/2)t+\sigma W_t}, \qquad t \ge 0,$$

where W_t is Brownian motion, $r \ge 0$, $\sigma \in \mathbb{R}$ constants. Its time average is

$$A(T) = \frac{1}{T}\int_0^T S(t)\,dt, \qquad T > 0.$$

Empirical discovery: $S(T)$ and $A(T)$ typically highly correlated – coefficient ≈ 0.85.
Problem: Calculating correlation coefficient is tricky.
Surprise: Divided differences occur naturally in the analysis, leading to great simplification and new insights from approximation theory.

Lévy and Cieselski subdivision-style construction of Brownian motion

Let $\{Z(q) : q \in \mathbb{Q}\}$ be independent normalized Gaussian random variables. Define $B(0) = 1$ and

$$B(k) = B(k-1) + Z(k), \qquad k = 1, 2, \ldots.$$

Then define

$$B\left(\frac{k+1/2}{2^n}\right) \\ = \frac{1}{2}\left(B\left(\frac{k}{2^n}\right) + B\left(\frac{k+1}{2^n}\right)\right) + 2^{-1-n/2} Z\left(\frac{k+1/2}{2^n}\right)$$

.

Now it was already known that

$$\mathbb{E}\left(A(T)^2\right)$$

is given by

$$\frac{2e^{(2r+\sigma^2)T}}{(r+\sigma^2)(2r+\sigma^2)T} + \frac{2}{rT^2}\left(\frac{1}{2r+\sigma^2} - \frac{e^{rT}}{r+\sigma^2}\right).$$

Surprise: This *is* a divided difference:

$$\mathbb{E}\left(A(T)^2\right) = 2\exp[0, rT, (2r+\sigma^2)T].$$

Key fact: $\mathbb{E}S(t) = e^{rt}$.
Simple link with divided differences:

$$
\begin{aligned}
\mathbb{E}A(T) &= \frac{1}{T}\int_0^T \mathbb{E}S(t)\,dt \\
&= \frac{e^{rT}-1}{rT} \\
&= \exp[0, rT].
\end{aligned}
$$

Coincidence? Let's try another.

We need a simple Lemma:

$$\mathbb{E}S(a)S(b) = e^{a(r+\sigma^2)}e^{br}, \qquad \text{for } 0 \le a \le b.$$

Proof: Straightforward Brownian motion exercise.
Then

$$\begin{aligned}
\mathbb{E}S(T)A(T) &= T^{-1}\int_0^T \mathbb{E}S(t)S(T)\,dt \\
&= T^{-1}\int_0^T e^{(r+\sigma^2)t}e^{rT}\,dt \\
&= \frac{e^{(2r+\sigma^2)T} - e^{rT}}{(r+\sigma^2)T} \\
&= \exp[rT, (2r+\sigma^2)T].
\end{aligned}$$

Similarly

$$
\begin{aligned}
\mathbb{E}(A(T)^2) &= T^{-2}\int_0^T \Big(\int_0^T \mathbb{E}S(t_1)S(t_2)\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T \Big(\int_0^{t_1} \mathbb{E}S(t_1)S(t_2)\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T \Big(\int_0^{t_1} e^{r(t_1+t_2)}e^{\sigma^2 t_2}\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T e^{rt_1}\Big(\frac{e^{(r+\sigma^2)t_1}-1}{r+\sigma^2}\Big)\,dt_1 \\
&= \frac{2}{(r+\sigma^2)T}\Big[\exp[0,(2r+\sigma^2)T]-\exp[0,rT]\Big] \\
&= 2\exp[0,rT,(2r+\sigma^2)T],
\end{aligned}
$$

Now we *expect* to see divided differences:

$$\begin{aligned}
&\mathbb{E}S(T)A(T) - \mathbb{E}S(T)\mathbb{E}A(T) \\
&\quad = \exp[rT, (2r+\sigma^2)T] - e^{rT}(e^{rT} - 1)/(rT) \\
&\quad = \exp[rT, (2r+\sigma^2)T] - \exp[rT, 2rT] \\
&\quad = \sigma^2 T \exp[rT, 2rT, (2r+\sigma^2)T],
\end{aligned}$$

and for the variance

$$\begin{aligned}
&\mathbb{V}S(T) \\
&\quad = \mathbb{E}(S(T)^2) - (\mathbb{E}S(T))^2 \\
&\quad = e^{(2r+\sigma^2)T} - e^{2rT} \\
&\quad = \sigma^2 T \exp[2rT, (2r+\sigma^2)T].
\end{aligned}$$

Finally, the correlation coefficient R is given by

$$\frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}.$$

Two obvious questions arise:

- Why do these iterated integrals lead to divided differences?
- So what?

Hermite–Genocchi

Let $f \in C^{(n)}(\mathbb{R})$ and let $a_0, a_1, \ldots, a_n$ be real numbers. Then

$$\begin{aligned} & f[a_0, a_1, \ldots, a_n] \\ &= \int_{S_n} f^{(n)}(t_0 a_0 + t_1 a_1 + \cdots + t_n a_n)\, dt_1 \cdots dt_n, \\ &= \int_0^1 dt_1 \cdots \int_0^{1-\sum_{k=1}^{n-1} t_k} dt_n\ f^{(n)}(\sum_{k=0}^{n} t_k a_k) \end{aligned}$$

integrating over the simplex

$$S_n = \{t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}_+^n : \sum_{k=1}^{n} t_k \le 1\}$$

and

$$t_0 = 1 - \sum_{k=1}^{n} t_k.$$

For the exponential function,

$$\exp[a_0, \ldots, a_n] = \int_{S_n} \exp(\sum_{k=0}^{n} t_k a_k)\, dt_1 \cdots dt_n.$$

For any nonsingular matrix

$$V = (v_1 \quad \cdots \quad v_n) \in \mathbb{R}^{n \times n},$$

let

$$K(V) = \mathrm{conv}\{0, v_1, \ldots, v_n\}.$$

Then

$$\frac{1}{|\det V|} \int_{K(V)} \exp(a^T y)\, dy$$

is equal to

$$\exp[0, (V^T a)_1, \ldots, (V^T a)_n].$$

[$(V^T a)_j$ is jth component of $V^T a$.]

If

$$V = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

then

$$\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 \exp\left(\sum_{k=1}^{n} a_k x_k\right)$$
$$= \exp[0, a_n, a_n + a_{n-1}, \ldots, a_n + a_{n-1} + \cdots + a_1].$$

Now we can compute higher moments of $A(T)$. We obtain

$$\mathbb{E}\left(A(T)^m\right) = m! \exp[b_0 T, b_1 T, \ldots, b_m T],$$

where

$$b_k = rk + \sigma^2 k(k-1)/2, \quad k \geq 0.$$

So what? Divided differences allow us to use the rich analytic toolbox of approximation theory:

- If $r = \sigma^2$, then the correlation coefficient $R = \sqrt{3}/2 = 0.866\ldots$
- Theorem[B and Fretwell] For any $r \geq 0$ and σ, the correlation coefficient satisfies $R \geq \frac{1}{\sqrt{2}} = 0.7071\ldots$

Thus the time-average is a remarkably good predictor for asset's price in the geometric Brownian motion universe.

In fact the correlation coefficient inequality is a special case of the following

Theorem Let $h \geq 0$ and define

$$E_n(x) = \exp[0, -h, -2h, \ldots, -nh, x], \qquad x \in \mathbb{R},\ n \geq 0.$$

Then $(E_n(x))$ is a log-concave sequence, i.e.

$$E_{n+1}(x)E_{n-1}(x) \leq E_n(x)^2, \qquad \text{for } n \geq 1.$$

Log-concave sequences: Enormous literature. See, e.g., Wilf, *Generatingfunctionology*.

Special Case: Define

$$R_m(\alpha) = e^\alpha - \sum_{k=0}^{m} \frac{\alpha^k}{k!},$$

for non-negative integer m and $\alpha \in \mathbb{R}$. Thus $R_m(\alpha)$ is the Taylor remainder (after $m+1$ terms) for the exponential function. Further

$$R_m(\alpha) = \alpha^{m+1} \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha].$$

Furthermore,

$$R_m'(\alpha) = R_{m-1}(\alpha), \qquad \text{for } m \ge 1, \alpha \in \mathbb{R}.$$

Lemma The exponential function Taylor remainders satisfy

$$\frac{R_{m+1}(\alpha)}{R_m(\alpha)} = 1 - \frac{1}{(m+1)!\exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha]}.$$

Proof

$$\begin{aligned} 1 - \frac{R_{m+1}(\alpha)}{R_m(\alpha)} &= \frac{R_m(\alpha) - R_{m+1}(\alpha)}{R_m(\alpha)} \\ &= \frac{p_{m+1}(\alpha) - p_m(\alpha)}{R_m(\alpha)} \\ &= \frac{\alpha^{m+1}}{(m+1)! R_m(\alpha)} \\ &= \frac{1}{(m+1)!\exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha]}. \end{aligned}$$

However $\alpha \mapsto \exp[\underbrace{0, 0, \dots, 0}_{m+1}, \alpha]$ is an increasing function, with derivative

$$\exp[\underbrace{0, 0, \dots, 0}_{m+1}, \alpha, \alpha].$$

Corollary $R_{m+1}(\alpha)/R_m(\alpha)$ is an increasing function.

Proof

$$\frac{d}{d\alpha}\frac{R_{m+1}(\alpha)}{R_m(\alpha)} = \frac{\exp[\overbrace{0, 0, \dots, 0}^{m+1}, \alpha, \alpha]}{(m+1)!\exp[\underbrace{0, 0, \dots, 0}_{m+1}, \alpha]^2}.$$

Hence

$$R_m(\alpha)^2 \geq R_{m+1}(\alpha)R_{m-1}(\alpha), \qquad \text{for } m \geq 1 \text{ and } \alpha \in \mathbb{R}.$$

because

$$0 \leq \frac{d}{d\alpha}\frac{R_{m+1}(\alpha)}{R_m(\alpha)} = \frac{R_m(\alpha)^2 - R_{m+1}(\alpha)R_{m-1}(\alpha)}{R_m(\alpha)^2}.$$

Now, when $h = 0$,

$$R_m(\alpha) = E_m(\alpha)\alpha^{m+1},$$

so $(E_m(\alpha))$ is also log-concave, i.e.

$$\exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha] \exp[\underbrace{0, 0, \ldots, 0}_{m-1}, \alpha] \le \exp[\underbrace{0, 0, \ldots, 0}_{m}, \alpha]^2.$$

Is it only true for exponentials?
Maple experiments show that

$$f[0, h, 2h, \ldots, nh], \qquad h > 0,$$

is a log-concave sequence for many (all?) completely monotonic functions, i.e. $(-1)^n f^{(n)}(x) \geq 0$, $x \geq 0$.
Bernstein–Widder Theorem: $f : [0, \infty) \to \mathbb{R}$ is completely monotonic if and only if

$$f(x) = \int_0^\infty e^{-xs}\, d\mu(s), \qquad x \geq 0,$$

for some positive Borel measure μ on $[0, \infty)$.

Let X be a Lévy-Stable process. Then the natural logarithm of its characteristic function is given by n

$$\ln \mathbb{E}[e^{iX\theta}] = \begin{cases} -\kappa^\alpha |\theta|^\alpha (1 - i\beta(\mathrm{sign}\ \theta) \tan \frac{\alpha\pi}{2}) + im\theta & \text{if } \alpha \neq 1 \\ -\kappa|\theta|(1 + i\beta\frac{2}{\pi}(\mathrm{sign}\ \theta) \ln|\theta|) + im\theta & \text{if } \alpha = 1 \end{cases}$$

where $\alpha \in (0, 2]$, $\kappa > 0$, and $\beta \in [-1, 1]$; we write $X \sim S_\alpha(\kappa, \beta, m)$

Then

$$S(T) = S(t) \exp\left((r+\mu)(T-t) - \sigma X_{T-t}\right),$$

where $X_{T-t} \sim S_\alpha\left((T-t)^{1/\alpha}, 1, 0\right)$. For risk-neutrality, $\mu = \sigma^\alpha \sec(\alpha\pi/2)$.
The correlation coefficient satisfies

$$R = \frac{\exp[rT, 2rT, (2r + \mu(2-2^\alpha))T]}{\sqrt{2\exp[2rT, (2r+\mu(2-2^\alpha))T]\exp[0, rT, 2rT, (2r+\mu(2-2^\alpha))T]}}$$

Theorem $R \geq 1/\sqrt{2}$.