ME: Large and Small

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You can download these notes from my office server

http://econ109.econ.bbk.ac.uk/brad/ME/Lectures/

and my home server

http://cato.tzo.com/brad/teaching/ME/

Theorem

1. *If two cubes have edges in the ratio $1 : L$, their surface areas are in the ratio $1 : L^2$ and their volumes in the ratio $1 : L^3$.*
2. *If two shapes are geometrically similar and if the distance between two points on one of them is to the distance between the two corresponding points on the second in the ratio $1 : L$, then corresponding areas are in the ratio $1 : L^2$ and corresponding volumes in the ratio $1 : L^3$.*

Only consider sets of geometrically similar animals (not necessarily the same species): each can be described by a single length L, e.g. the length of a corresponding bone or the total height.

Theorem

1. *The strength of a bone is proportional to its cross-sectional area and hence to* L^2.
2. *The total weight is proportional to the volume and hence to* L^3.

Example

If animal A is three times as high as B, then its bones can withstand 9 times the weight, but its weight has increased by a factor of 27. What happens?

Ultimately L^3 beats L^2, so there is a maximum animal size for a given geometric shape.

Power output varies as L^2 for three reasons.

1. Rate of heat loss: Muscles are devices for turning chemical into mechanical energy and their efficiency is about 25%; the remaining 75% of the energy is waste heat. This heat is lost through the surface, so its dissipation rate is proportional to L^2. Hence average power output exceeding L^2 would lead to overheating.
2. Oxygen supply: The volume of blood pumped by the heart per unit time is proportional to L^2 too.
3. Physical strength: If the force applied by the foot is T and the body moves forward a distance d, then the speed v is given by conservation of energy

$$Td = \frac{1}{2}mv^2.$$

Now $T \propto L^2$ and $d \propto L$, so the LHS is proportional to L^3. The mass $m \propto L^3$ too, so the speed v does not depend on L. If t is the time taken for muscle contraction, then $t \propto L/v \propto L$, so the power output is $Td/t \propto L^2$.

Example (Running on the flat)

If an animal is moving at speed v, then the force opposing its motion, or **drag** is proportional to its surface area and the square of its speed, i.e. drag $\propto L^2 v^2$, so the power required to overcome air resistance is $\propto L^2 v^3$. The power output is proportional to L^2, so all animals of a given shape should have the same top speed.

In general, the drag force in a fluid (i.e. a liquid or gas) is

$$\text{drag} = \frac{1}{2} c_D \rho A v^2$$

where ρ is the density of the fluid and c_D is the **drag coefficient**: a dimensionless number used to quantify drag. For example, $c_D = 0.82$ for a long cylinder, $c_D \approx 0.4$ for an SUV, but a streamlined car can achieve $c_D \approx 0.2$.

Thus the power needed to push an object through as fluid increases as the cube of the velocity.

Example

A car travelling at 80 km/h might need only 7.5 kW to overcome drag, but the same car at 160 km/h needs 60 kW. [For historical reasons, car power is still given in horsepower in the US and UK, a unit popularised by James Watt to convey the usefulness of his steam engines: 10 horsepower is approximately 7.5 kW.

Terminal Velocity The drag force in a fluid at **high speed** v is

$$F_{\text{drag}} = \frac{1}{2} c_D \rho A v^2$$

where ρ is the density of the fluid and c_D is the **drag coefficient**. When F_{drag} is equal to the weight mg, the body no longer accelerates and we have reached the **terminal velocity**

$$V_T = \sqrt{\frac{mg}{\frac{1}{2} c_D \rho A}} = \sqrt{\frac{2mg}{c_D \rho A}}$$

ignoring buoyancy effects (relevant in water for mammals, but not in air). For humans, V_T is roughly 50 m/s, or 180 km/h, so fatal with high probability.
At **low speed** v, we find that the terminal velocity $V_T \propto L$, because air resistance at low speed v is proportional to the area ($\propto L^2$) multiplied by v (**not** v^2), while the weight $mg \propto L^3$.

Example (Maximum survivable velocity)

The kinetic energy at velocity v is $\frac{1}{2}mv^2$. If the body comes to rest over a short fixed distance Δ, then the force F is proportional to the kinetic energy, i.e.

$$F = \frac{1}{2}mv^2/\Delta = \propto L^3 v^2.$$

The maximum force we can tolerate $F \propto L^2$, so we must have $v^2 \propto L^{-1}$, i.e. the maximum survivable velocity

$$v \propto \frac{1}{L^{1/2}}.$$

Example (Running uphill)

If the rate of increase in height is v, then work is done against gravity at a rate $\propto L^3 v$. Since the power output is $\propto L^2$, we must have $v \propto L^{-1}$.

Thus horses walk slowly uphill, humans walk, while a dog can run.

General flight requires aerodynamics beyond the scope of this course, but we can deal with hovering: a bird produces a downward jet of air and the momentum of this jet per unit time equals the lift generated, and hence the weight of the bird.
Let the jet have velocity v and cross-sectional area A. The mass of air projected downward in unit time is $\rho A v$, where ρ is the air density. If the bird has mass m, then

$$mg = \rho A v^2.$$

But $m \propto L^3$, $A \propto L^2$ (for birds, but not small insects), so $v \propto L^{1/2}$. The power output P generating the jet is the kinetic energy in the jet per unit time, i.e.

$$P \propto \frac{1}{2}\rho A v \times v^2 \propto L^{3.5}.$$

We know that $P \propto L^2$, so there is an upper limit to the size of birds that can hover, which is about 20 kg.