ME: Large and Small Extended

Brad Baxter
Birkbeck College, University of London

January 31, 2024

You can download these notes from my office server

`http://econ109.econ.bbk.ac.uk/brad/teaching/ME/`

and my home server

`http://cato.tzo.com/brad/teaching/ME/`

Theorem

1. *If two cubes have edges in the ratio $1 : L$, their surface areas are in the ratio $1 : L^2$ and their volumes in the ratio $1 : L^3$.*
2. *If two shapes are geometrically similar and if the distance between two points on one of them is to the distance between the two corresponding points on the second in the ratio $1 : L$, then corresponding areas are in the ratio $1 : L^2$ and corresponding volumes in the ratio $1 : L^3$.*

Only consider sets of geometrically similar animals (not necessarily the same species): each can be described by a single length L, e.g. the length of a corresponding bone or the total height.

Theorem

1. *The strength of a bone is proportional to its cross-sectional area and hence to L^2.*
2. *The total weight is proportional to the volume and hence to L^3.*

Example

If animal A is three times as high as B, then its bones can withstand 9 times the weight, but its weight has increased by a factor of 27. What happens?

Ultimately L^3 beats L^2, so there is a maximum animal size for a given geometric shape.

Application: BMI

The **Body Mass Index (BMI)** is defined to be the mass divided by the square of the height. Thus a male weighing 100 kg whose height is 1.83 m has a BMI of $100/1.83^2 = 29.86\ \mathrm{kg}/\mathrm{m}^2$.
The mass is proportional to the weight and the bone area is proportional to L^2, so BMI is proportional to bone stress in N/m^2.
An adult whose BMI is less than 18.5 kg/m^2 is considered underweight, while the above example is considered overweight.

BMI flaws: Humans are only approximately geometrically similar; for example, there are examples of healthy athletes whose BMI places them in the obese category because of very high muscle mass.

Power output varies as L^2 for three reasons.

1. Rate of heat loss: Muscles are devices for turning chemical into mechanical energy and their efficiency is about 25%; the remaining 75% of the energy is waste heat. This heat is lost through the surface, so its dissipation rate is proportional to L^2. Hence average power output exceeding L^2 would lead to overheating.
2. Oxygen supply: The volume of blood pumped by the heart per unit time is proportional to L^2 too.
3. Physical strength: If the force applied by the foot is T and the body moves forward a distance d, then the speed v is given by conservation of energy
$$Td = \frac{1}{2}mv^2.$$
Now $T \propto L^2$ and $d \propto L$, so the LHS is proportional to L^3. The mass $m \propto L^3$ too, so the speed v does not depend on L. If t is the time taken for muscle contraction, then $t \propto L/v \propto L$, so the power output is $Td/t \propto L^2$.

Kinetic Energy (KE)

KE was not known to Newton but was derived from Newton's laws of motion by French physicist and mathematician *Émilie du Châtelet* in the early 18th century.

We imagine one dimensional movement along a line caused by a force $F(x)$ that depends on the position x. By Newton's Second Law,

$$m\frac{d^2x}{dt^2} = F(x).$$

Trick: Let $v = dx/dt$ and use

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}.$$

Thus we have the differential equation

$$mv\frac{dv}{dx} = F(x)$$

or

$$\frac{d}{dx}\left(\frac{1}{2}mv^2\right) = F(x).$$

Integrating from position x_1 to position x_2, we find

$$\frac{1}{2}mv(x_2)^2 - \frac{1}{2}mv(x_1)^2 = \int_{x_1}^{x_2} F(x)\,dx.$$

If the force is constant, i.e. $F(x) \equiv F$, then

$$\frac{1}{2}mv(x_2)^2 - \frac{1}{2}mv(x_1)^2 = F(x_2 - x_1).$$

Example (Running on the flat)

If an animal is moving at speed v, then the force opposing its motion, or **drag** is proportional to its surface area and the square of its speed, i.e. drag $\propto L^2 v^2$, so the power required to overcome air resistance is $\propto L^2 v^3$. The power output is proportional to L^2, so all animals of a given shape should have the same top speed.

In general, the drag force in a fluid (i.e. a liquid or gas) is

$$\text{drag} = \frac{1}{2} c_D \rho A v^2$$

where ρ is the density of the fluid and c_D is the **drag coefficient**: a dimensionless number used to quantify drag. For example, $c_D = 0.82$ for a long cylinder, $c_D \approx 0.4$ for an SUV, but a streamlined car can achieve $c_D \approx 0.2$.

Deriving the drag force Imagine a cuboidal car with cross-sectional area A moving at speed v through still air. Equivalently, imagine the car being stationary and the wind moving at speed v towards the car. In one second, a cuboid of air of length v and cross-sectional area A hits the front of the car. The mass of air $M = \rho A v$, where ρ is the air's density, and so the KE of the air hitting the car every second is given by

$$\frac{1}{2}mv^2 = \frac{1}{2}\rho A v^3.$$

To a first approximation, this is proportional to the power needed for the car to move at speed v. The constant of proportionality c_D is called the **drag coefficient**, i.e.

$$\text{power required to move at } v = \frac{1}{2}c_D \rho A v^3.$$

By conservation of energy, this must equal the work done per second, which is $\mathrm{drag} \times v$, so

$$\mathrm{drag} \times v = \frac{1}{2} c_D \rho A v^3$$

or

$$\mathrm{drag} = \frac{1}{2} c_D \rho A v^2.$$

Braking distance and harm to humans If a car is travelling at speed v and its braking force F is constant, then its braking distance d satisfies

$$Fd = \frac{1}{2}mv^2,$$

where m is the mass of the car. In other words, the braking distance increases quadratically with speed v.

Exercise

The typical braking distance at $40\ \mathrm{mph}$ *is* $24m$*. What are the typical braking distances at* $60\ \mathrm{mph}$ *and* $70\ \mathrm{mph}$*?*

Example

The harm done in collision increases with the KE, so reducing urban speeds from, say, $30\ \mathrm{mph}$ to $20\ \mathrm{mph}$ reduces KE by roughly 55%.

Thus the power needed to push an object through the fluid increases as the **cube** of the velocity.

Example

A car travelling at 80 km/h might need only 7.5 kW to overcome drag, but the same car at 160 km/h needs 60 kW. [For historical reasons, car power is still given in horsepower in the US and UK, a unit popularised by James Watt to convey the usefulness of his steam engines: 10 horsepower is approximately 7.5 kW.]

Exercise

Suppose a country reduces the motorway speed limit from 80 mph *to* 60 mph*. By what percentage will motorway fuel consumption reduce, assuming that all cars drive at the maximum speed?*

Terminal Velocity The drag force in a fluid at **high speed** v is

$$F_{\text{drag}} = \frac{1}{2} c_D \rho A v^2$$

where ρ is the density of the fluid and c_D is the **drag coefficient**. When F_{drag} is equal to the weight mg, the body no longer accelerates and we have reached the **terminal velocity**

$$V_T = \sqrt{\frac{mg}{\frac{1}{2} c_D \rho A}} = \sqrt{\frac{2mg}{c_D \rho A}}$$

ignoring buoyancy effects (relevant in water for mammals, but not in air). For humans, V_T is roughly 50 m/s, or 180 km/h, so fatal with high probability.

Thus

$$V_T \propto \sqrt{m/A} \propto L^{1/2}$$

and the KE at terminal velocity is

$$\frac{1}{2} m V_T^2 \propto L^4.$$

It's the KE at collision that harms or kills.

In Haldane's words: You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away. A rat would probably be killed, though it can fall safely from the eleventh story of a building; a man is killed, a horse splashes.

Example (Running uphill)

If the rate of increase in height is v, then work is done against gravity at a rate $\propto L^3 v$. Since the power output is $\propto L^2$, we must have $v \propto L^{-1}$.

Thus horses walk slowly uphill, humans walk, while a dog can run.

General flight requires aerodynamics beyond the scope of this course, but we can deal with **hovering**: a bird produces a downward jet of air and the momentum of this jet per unit time equals the lift generated, and hence the weight of the bird.
Let the jet have velocity v and cross-sectional area A. The mass of air projected downward in unit time is $\rho A v$, where ρ is the air density. If the bird has mass m and hovers, then

$$mg = \rho A v^2.$$

But $m \propto L^3$, $A \propto L^2$ (for birds, but not small insects), so $v \propto L^{1/2}$. The power output P generating the jet is the kinetic energy in the jet per unit time, i.e.

$$P \propto \frac{1}{2}\rho A v \times v^2 \propto L^{3.5}.$$

We know that $P \propto L^2$, so there is an upper limit to the size of birds that can hover, which is about 20 kg.

Earlier stated that an animal's power output is proportional to L^2: $P \propto L^2$. Thus, for mass $M \propto L^3$ we expect

$$P = CM^{\alpha}$$

where $\alpha = 2/3$, or

$$\log P = \log C + \alpha \log M.$$

Problem: Biologists have computed α for many animals and find $\alpha \approx 3/4$, **not** $2/3$. This is sometimes called **Kleiber's Law**, first noted in 1932. You can find many references online, but the next graph is from Kleiber (1932): "Body size and metabolism", *Hilgardia*. **6**:315–353:

`https://hilgardia.ucanr.edu/Abstract/?a=hilg.v06n11p315`

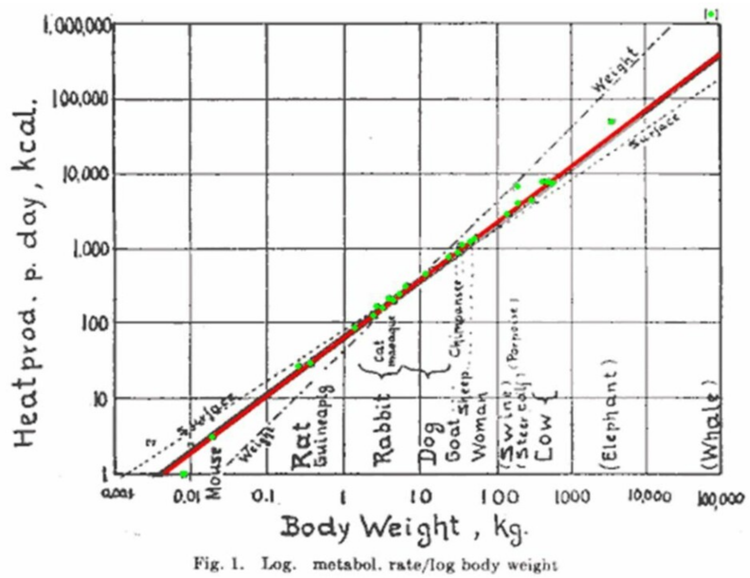

Figure: Kleiber's Law from Kleiber (1932)

Kleiber's Law: Really it's only an approximation, but the key point seems to be that lungs are **fractal**: their area does not increase as L^2. Still, the difference is small.

Example (Mice and Elephants)

The mass ratio for elphants and mice is roughly

$$\frac{M_E}{M_M} = 10^5.$$

Then the power ratio for the $2/3$ power law is

$$R_{2/3} = \left(\frac{M_E}{M_M}\right)^{2/3} = 10^{10/3}$$

but for Kleiber's $3/4$ power law we find

$$R_{3/4} = \left(\frac{M_E}{M_M}\right)^{3/4} = 10^{15/4}.$$

Exercise

Show that

$$\frac{R_{3/4}}{R_{2/3}} = 10^{5/12} \approx 2.6.$$

Kleiber's Law conclusion: Many textbooks on mathematical biology still use $P \propto L^2$ because the error is small, the insight is useful and Kleiber's law is imperfect too. Kleiber's law is used by farmers to estimate food needs for animals.

There are several books that deal with these topics very clearly and from which I have learnt much. One particularly excellent book is

The Pleasures of Counting, by T. W. Körner, Cambridge University Press, 1996.

You might also be interested in reading
On Being the Right Size, by J. B. S. Haldane, Oxford University Press, 1985.