ME: Sound and Music

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You can download these notes from my office server

`http://econ109.econ.bbk.ac.uk/brad/teaching/ME/`

and my home server

`http://cato.tzo.com/brad/teaching/ME/`

Music is old: all dates Before Present (BP)

100000 BP: anatomically modern humans

50000 BP: Exponential increase in human artefacts

45000 BP: First musical instruments

12000 BP: End of last Ice Age

10000 BP: Farming begins in Middle East

7000 BP: Indo-European language speakers spread in Europe

4000 BP: Use of bronze begins

3400 BP: First music notation in Mesopotamia

3000 BP: Use of Iron (and Late Bronze Age Collapse)

3000 BP: Phoenicians (Lebanon) invent money and the alphabet

2800 BP: Greeks become literate using modified Phoenician alphabet. Romans modify Greek alphabet shortly after.
2600 BP: First Greek mathematics
1000 BP: First modern music notation (Guido d'Arezzo)
700 BP: Chords, counterpoint and keyboard instruments
300 BP: Modern physics of music begins

Our ears detect changes in air pressure: sound.
The greater the variation in pressure, the louder the sound.

The sound made by a musical instrument repeats, perceived as a note at a given pitch.

The frequency of repetition determines how low or high the corresponding pitch: the higher the frequency, the higher the pitch.

Frequencies are usually measured in Hertz (Hz), or repetitions per second.

Frequency and Wavelength

For sound moving in air at $v = 330\ \mathrm{m/s}$, the frequency $f\ \mathrm{Hz}$ and the wavelength $\lambda\ \mathrm{m}$ are related by

$$v = f\lambda.$$

Example

If $f = 256\ \mathrm{Hz}$ (middle C), then the wavelength

$$\lambda = v/f = 1.2891\ \mathrm{m}.$$

The lowest bass note on an organ is typically $f = 64\ \mathrm{Hz}$, so the wavelength

$$\lambda = v/f = 5.1562\ \mathrm{m}.$$

Thus the longest pipes on an organ are for the bass notes, while the shortest are for the high notes.

Exercise

Find the frequency of the note produced by a 64 foot organ pipe.

Doppler effect: If a sound source moves towards us (e.g. a siren) the sound waves arrive closer together: lower wavelength so higher frequency: we perceive the sound increasing in pitch.

When the source moves away, the reverse happens: the waves are stretched out, i.e. the wavelength increases and the frequency decreases, so the sound becomes lower in pitch.

Pythagoras (possibly) conducted experiments plucking strings of different lengths on a monochord (a simple instrument with one string).
Key discovery: If you pluck a string, and then halve the length and pluck it again, the two sounds produced are very similar and sound pleasant together. Strings two-thirds or three-quarters as long as the original also make pleasing harmonies with the original string.
Wavelength: The frequency is inversely proportional to string length, so halving the string length has the effect of doubling the frequency.

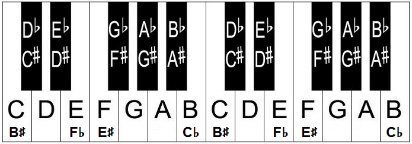
Better to discuss relative frequencies of tones rather than string lengths.

The two other important intervals are obtained from ratios 3:2 (the shorter string is two-thirds the length and the frequency increases by a factor of $3/2$) and 4:3 (the shorter string is three-quarters the length, raising the frequency by a factor of $4/3$). The doubling of the frequency produces a sound so similar that we almost think of it as the same note. The interval between the two notes is called an **octave**.

Example: Typically women sing “the same note” an octave higher than men. But what about intervals smaller than the octave? How is the octave divided? Why is it called an octave?

Dividing the Octave

The Pythagoreans prioritised what are now called octaves (2:1), perfect fifths (3:2) and perfect fourths (4:3). If we follow a perfect fifth with a perfect fourth, the net effect is to multiply these ratios. Thus, if our initial frequency is f, then a perfect fifth above this would have frequency $\frac{3}{2}f$ and a fourth above that would be $\frac{4}{3} \times \frac{3}{2} \times f = 2f$, an octave higher. However, there are other intervals, and in order to discuss them we need to learn some simple musical notation.

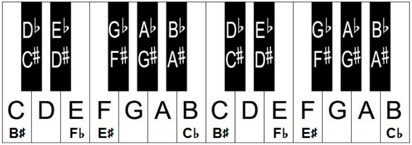


Here is a keyboard. The white notes on a piano are labelled A to G. These represent the notes in the scale of C major: we have C, D, E, F, G, A, B and then the eighth note is another C, hence the word **octave**.

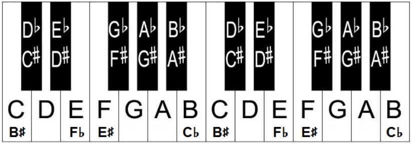
Because G is the fifth note, the interval from C to G is called a **perfect 5th**.

From G up to C is a **fourth**, which corresponds to 4:3.

Pairs of notes an integer number of octaves apart have the same letter name.



The black note between two white notes is the **sharp** (♯) of the note below, and the **flat** (♭) of the note above. So, the note between C and D can be referred to as C♯ or D♭. The smallest gap is a **semitone**, such as that between E and F, or F and F♯. Two semitones make a **tone**. An **octave** is made up of twelve semitones, giving one note for each of the twelve different pitch classes of modern Western music.



Some history: The keyboard was probably invented in the 13th century, probably in the Low Countries (modern Belgium and the Netherlands).

The first keyboards only had white notes, so C to C really was 8 notes, hence the name octave.

The first black note added was B♭, initially denoted ♭, and then the notation was extended: it's a mess! The note B became the symbols ♮ and ♯. In German musical notation B became H! The terms flat and sharp refer to the string tension (these are called soft and hard in some languages).

Every major scale has the same arrangement of notes and intervals between them. For instance, in C major, the intervals between E and F, and between B and C, are semitones; the remaining intervals are tones. The pattern

tone, tone, semitone, tone, tone, tone, semitone

is replicated for every starting (**tonic**) note.

Example

G major is G, A, B, C, D, E, F$\sharp$, G.

Exercise

Find the notes for D major and F major.

On a modern keyboard, we can obtain all twelve different pitch classes using intervals of a fifth:

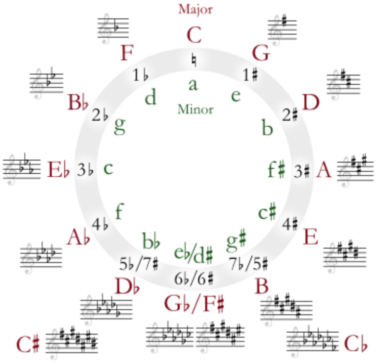
$$\mathbf{C, G, D, A, E, B, F\sharp, C\sharp, G\sharp, D\sharp, A\sharp, F, C}$$

We return to C seven octaves above.

Usually display the sequence of fifths

C, G, D, A, E, B, F♯, C♯, G♯, D♯, A♯, F, C

as the Circle of Fifths:



Pythagoras legend: Pythagoras is said to have experimented with progressions of perfect fifths and octaves on two monochords.
M1: He started with a long string that produced a low note. He then successively halved the length of the string 7 times, to end up with a tone 7 octaves higher.

M2: He started with the same low note, but shortened the string length by 2/3 each time, increasing the frequency by a factor of 3/2: raising the pitch by a perfect fifth.
Discovery: After 12 steps, the final notes were sounded together but are **NOT** identical: they are unpleasantly out of tune.

Why? If we start with a note at frequency f, and raise the pitch by seven octaves, the new frequency is $2^7 f = 128f$. But if instead we raise the pitch by twelve perfect fifths, the new frequency is

$$(3/2)^{12} f = 129.746337890625 0f.$$

On a modern keyboard this problem does not exist: the ‘fifths’ on a modern keyboard **not** true fifths. We call them **tempered fifths**.

Jargon: When necessary, we say that the interval made by the ratio 3:2 is a “pure” perfect fifth, or just a pure fifth. Similarly, 4:3 is a pure fourth; later we will encounter other pure intervals.

Does it matter? Listen to the example of mean tone I’ve provided to hear the problem!

Subdividing the octave: Cents

Alexander John Ellis (1814–1890), invented a fine subdivision of the octave known as the cent. It has become the standard method of representing and comparing musical pitches and intervals. This compares frequency ratios using a logarithmic measure in which one octave is divided into 1200 cents. One advantage of this is that it converts multiplication to addition. The definition is that the ratio $\frac{f_2}{f_1}$ of two frequencies is equal to c cents, where

$$\frac{f_2}{f_1} = 2^{c/1200}.$$

A frequency ratio r therefore corresponds to $1200 \log_2 r$ cents. If we want to follow a ratio r_1 that is a_1 cents, with a ratio r_2 that is a_2 cents, then the ratio of the outcome is $r_1 r_2$, but the new cent value is

$$1200 \log_2(r_1 r_2) = 1200(\log_2(r_1) + \log_2(r_2)) = 1200 \log_2(r_1) + 1200 \log_2(r_2)$$

A major advantage of this notation is that the same interval at different frequencies comes out at the same number of cents, and this tallies with how the human ear differentiates frequencies. Most people find that the smallest discernible difference in pitch is about 5 cents.

Example

To calculate cent values on a calculator, recall that for any $x > 0$ we have

$$\log_2(x) = \frac{\log_{10}(x)}{\log_{10}(2)} = \frac{\ln x}{\ln 2}.$$

The difference between 7 octaves $(128f)$ and 12 pure fifths $(\left(\frac{3}{2}\right)^{12} f)$ corresponds to a ratio of $r = \frac{\left(\frac{3}{2}\right)^{12} f}{128f}$. The cent value is given by

$$1200 \log_2(r) = \frac{1200 * (12 * \ln(3/2) - \ln(128))}{\ln 2} \approx 23.46 \text{ cents}.$$

This is unpleasant!

Exercise

What is the cent value of a pure fifth (the $\frac{3}{2}$ ratio)?

Exercise

What is the cent value of a pure fourth (the $\frac{4}{3}$ ratio)?

Exercise

What is the cent value of a pure major third (the $\frac{5}{4}$ ratio)?

Tuning methods: For string instruments such as the violin, which has 4 strings, we can tune them in perfect fifths. This isn't practical on a keyboard: if we use exact perfect fifths to tune the instrument (Pythagorean tuning) then it sounds horrible!

Tempered fifths: Keyboard instruments became much more common (and less expensive) in the 16th century and a new tuning method was devised: mean tone. Here every fifth is "tempered" to $5^{1/4}$, which means that the sequence C, G, D, A, E produces an E with frequency exactly 5 times the frequency of C. If we reduce this by two octaves, then E has frequency $5/4$ times the frequency of C, which is a Pythagorean pure major third.