ME: Further Sound and Music

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`http://econ109.econ.bbk.ac.uk/brad/teaching/ME/`

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`http://cato.tzo.com/brad/teaching/ME/`

Pythagorean Tuning: We generate 12 fifths over 7 octaves

$$A^\flat - E^\flat - B^\flat - F - C - G - \mathrm{D} - A - E - B - F^\sharp - C^\sharp - G^\sharp.$$

using true perfect fifths for every interval except for the final one $C^\sharp - G^\sharp$, where we replace $G^\sharp$ by a copy of $A^\flat$ that is 7 octaves higher.

Hence the frequencies of the notes in Pythagorean tuning are given by

$$f, pf, p^2 f, \ldots, p^{11} f, p^{11} qf,$$

where $p = 3/2$ and $p^{11} q = 2^7$.

Pythagorean problems: We see that

$$q = \frac{2^7}{(3/2)^{11}} = \frac{2^{18}}{3^{11}} \approx 1.47981055.$$

This is a very flat Wolf interval! Comparing it to a true perfect fifth, we find

$$1200 \frac{\ln(q/1.5)}{\ln 2} \approx -23.46 \text{ cents.}$$

[Remember: ± 5 cents $\approx$ in tune.]

Pythagorean Thirds are bad: Each step of 4 perfect fifths increases the frequency from f to

$$(3/2)^4 f = 5.0625 f.$$

This would be exactly $5f$ for a true major third. The error is

$$1200 \frac{\ln((3/2)^4/5)}{\ln 2} \approx 21.50629 \text{ cents.}$$

Why does this matter? The invention of systematic harmony in the 13th century was based on major triads in the circle of fifths: if the base note (or **tonic**) is C, then the corresponding major triad is called **C Major**: $C - E - G$.
This fundamental major triad comes with two closely associated major triads from the two adjacent notes in the circle of fifths, i.e. G Major ($G - B - D$) and F Major ($F - A - C$). You might know this as the $1 - 4 - 5$ rule. These sound so unpleasant in Pythagorean tuning that music essentially abandoned Pythagorean tuning except as a historical curiosity.

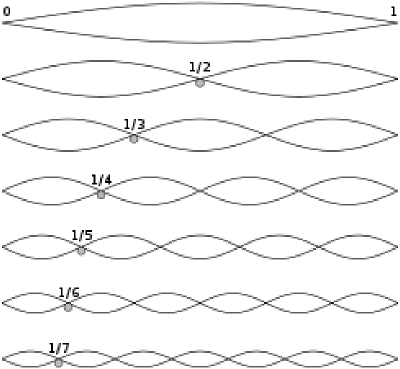
There are **minor triads** too: **C minor** is defined to be $C - E^\flat - G$.

Example

Every note belongs to exactly 3 major triads and 3 minor triads. For example, C belongs to the major triads $C - E - G$. $A^\flat - C - E^\flat$ and $F - A - C$ and the minor triads $C - E^\flat - G$, $A - C - E$ and $F - A^\flat - C$.

The Harmonic Series

A taut string fixed at both ends has the following resonances:



Intervals as rational numbers: the "Just" Scale

$$\begin{array}{lr} C - D & 9/8 \\ C - E^\flat & 6/5 \\ C - E & 5/4 \\ C - F & 4/3 \\ C - G & 3/2 \\ C - A & 5/3 \\ C - B & 15/8 \\ C - C' & 2 \end{array}$$

BUT cannot use these frequency ratios and use all keys and triads.

Instead we use temperaments: compromises that avoid major dissonances but approximate these intervals.

Meantone: This is the subject of HW2. We generate

$$A^\flat - E^\flat - B^\flat - F - C - G - \mathrm{D} - A - E - B - F^\sharp - C^\sharp - G^\sharp.$$

using the same tempered fifth for every interval except for the final one $C^\sharp - G^\sharp$, where we replace $G^\sharp$ by a copy of $A^\flat$ that is 7 octaves higher, and we choose the tempered fifth so that $C - E$ is a perfect major third. Hence the frequency ratios are

$$A^\flat \underset{r}{\rightarrow} E^\flat \underset{r}{\rightarrow} B^\flat \underset{r}{\rightarrow} F \underset{r}{\rightarrow} C \underset{r}{\rightarrow} G \underset{r}{\rightarrow} \mathrm{D} \underset{r}{\rightarrow} A \underset{r}{\rightarrow} E \underset{r}{\rightarrow} B \underset{r}{\rightarrow} F^\sharp \underset{r}{\rightarrow} C^\sharp \underset{W}{\rightarrow} G^\sharp,$$

where $r = 5^{1/4}$ and $r^{11}W = 2^7$.

Example

We have

$$1200\frac{\ln(r/1.5)}{\ln 2} \approx -5.37657 \text{ cents},$$

i.e. every meantone tempered fifth is slightly flat and it's audibly different from a true perfect fifth, but only just.

Exercise

This is the main HW2 problem: find W and

$$1200 \frac{\ln(W/1.5)}{\ln 2}.$$

Alternative Wolves: The defining equations of meantone are $r = 5^{1/4}$ and $r^{11} W = 2^7$ and here and in HW2 I have chosen the Wolf interval W to occur between $C^\sharp$ and $G^\sharp$. However, we can insert the Wolf interval between **any** two fifths, such as

$$A^\flat \underset{W}{\rightarrow} E^\flat \underset{r}{\rightarrow} B^\flat \underset{r}{\rightarrow} F \underset{r}{\rightarrow} C \underset{r}{\rightarrow} G \underset{r}{\rightarrow} \mathrm{D} \underset{r}{\rightarrow} A \underset{r}{\rightarrow} E \underset{r}{\rightarrow} B \underset{r}{\rightarrow} F^\sharp \underset{r}{\rightarrow} C^\sharp \underset{r}{\rightarrow} G^\sharp,$$

and $A^\flat - E^\flat$ is the usual choice for the Wolf interval.

KEY IDEA: Temper the fifths, i.e. no longer pure fifths.

A well-tempering: Kirnberger III: Here the 12 ratios are given by

$$A^\flat \underset{p}{\rightarrow} E^\flat \underset{p}{\rightarrow} B^\flat \underset{p}{\rightarrow} F \underset{p}{\rightarrow} C \underset{r}{\rightarrow} G \underset{r}{\rightarrow} \mathrm{D} \underset{r}{\rightarrow} A \underset{r}{\rightarrow} E \underset{p}{\rightarrow} B \underset{p}{\rightarrow} F^\sharp \underset{S}{\rightarrow} C^\sharp \underset{p}{\rightarrow} G^\sharp,$$

where $p = 3/2$, $r = 5^{1/4}$ and $p^7 r^4 S = 2^7$, i.e. $5p^7 S = 2^7$.

Exercise

Show that

$$S = \frac{2^{14}}{3^7 \cdot 5}$$

and that

$$1200 \frac{\ln(S/1.5)}{\ln 2} \approx -1.95372 \text{ cents.}$$

S is not a wolf interval ($|-1.95| \leq 5$ cents), one true major third ($C - E$), and 7 true perfect fifths. Kirnberger III is generally a good temperament for playing Bach (Kirnberger was one of Bach's pupils): it can play in any key and we can hear the difference between keys.

Another well-tempering: Werckmeister III: Here the 12 ratios are

$$A^\flat \underset{p}{\to} E^\flat \underset{p}{\to} B^\flat \underset{p}{\to} F \underset{p}{\to} C \underset{\alpha}{\to} G \underset{\alpha}{\to} \mathrm{D} \underset{\alpha}{\to} A \underset{p}{\to} E \underset{p}{\to} B \underset{\alpha}{\to} F^\sharp \underset{p}{\to} C^\sharp \underset{p}{\to} G^\sharp,$$

where $p = 3/2$ and $p^8\alpha^4 = 2^7$.

Werckmeister III is easy to learn to tune without modern electronic aids. In the 18th century, organs tuned in meantone were relatively easy to retune in Werckmeister III. It's another good temperament for Bach.

Yet another well-tempering: Vallotti: Here the 12 ratios are

$$A^\flat \underset{p}{\rightarrow} E^\flat \underset{p}{\rightarrow} B^\flat \underset{p}{\rightarrow} F \underset{\beta}{\rightarrow} C \underset{\beta}{\rightarrow} G \underset{\beta}{\rightarrow} \mathrm{D} \underset{\beta}{\rightarrow} A \underset{\beta}{\rightarrow} E \underset{\beta}{\rightarrow} B \underset{p}{\rightarrow} F^\sharp \underset{p}{\rightarrow} C^\sharp \underset{p}{\rightarrow} G^\sharp,$$

where $p = 3/2$ and $p^6\beta^6 = 2^7$. Vallotti is also good for Bach (my performance of the Goldberg Aria is played on a spinet in Vallotti).

Young II: This is almost identical to Vallotti.

$$A^\flat \underset{p}{\rightarrow} E^\flat \underset{p}{\rightarrow} B^\flat \underset{p}{\rightarrow} F \underset{p}{\rightarrow} C \underset{\beta}{\rightarrow} G \underset{\beta}{\rightarrow} \mathrm{D} \underset{\beta}{\rightarrow} A \underset{\beta}{\rightarrow} E \underset{\beta}{\rightarrow} B \underset{\beta}{\rightarrow} F^\sharp \underset{p}{\rightarrow} C^\sharp \underset{p}{\rightarrow} G^\sharp,$$

and, as for Vallotti, here $p = 3/2$ and $p^6\beta^6 = 2^7$.

Barnes: This is almost Vallotti and was created by John Barnes in the late 1970s, following a statistical analysis of the Well-Tempered Clavier.

$$A^\flat \underset{p}{\rightarrow} E^\flat \underset{p}{\rightarrow} B^\flat \underset{p}{\rightarrow} F \underset{\beta}{\rightarrow} C \underset{\beta}{\rightarrow} G \underset{\beta}{\rightarrow} \mathrm{D} \underset{\beta}{\rightarrow} A \underset{\beta}{\rightarrow} E \underset{p}{\rightarrow} B \underset{\beta}{\rightarrow} F^\sharp \underset{p}{\rightarrow} C^\sharp \underset{p}{\rightarrow} G^\sharp,$$

and, as for Vallotti, here $p = 3/2$ and $p^6\beta^6 = 2^7$.

Equal Temperament (ET): Here the frequencies of the 12 notes starting with C at frequency f are given by $2^{k/12}f$, for $0 \le k \le 12$. In other words, we have

$$A^\flat \underset{\tau}{\rightarrow} E^\flat \underset{\tau}{\rightarrow} B^\flat \underset{\tau}{\rightarrow} F \underset{\tau}{\rightarrow} C \underset{\tau}{\rightarrow} G \underset{\tau}{\rightarrow} \mathrm{D} \underset{\tau}{\rightarrow} A \underset{\tau}{\rightarrow} E \underset{\tau}{\rightarrow} B \underset{\tau}{\rightarrow} F^\sharp \underset{\tau}{\rightarrow} C^\sharp \underset{\tau}{\rightarrow} G^\sharp,$$

where $\tau = 2^{7/12}$.

History: Zhu Zaiyu (1584) and Simon Stevin (1585) independently discovered ET. Vincenzo Galilei also promoted ET in the 16th century.

Example

The ET fifth is $2^{7/12} \approx 1.4983$. The difference between this and a pure perfect fifth is given by

$$1200 \frac{\ln(2^{7/12}/1.5)}{\ln 2} \approx -1.9550 \text{ cents.}$$

This is close but slightly flat, but the major third $2^{4/12} = 2^{1/3}$ is unpleasantly sharp:

$$1200 \frac{\ln(2^{1.3}/1.25)}{\ln 2} \approx 13.686 \text{ cents.}$$

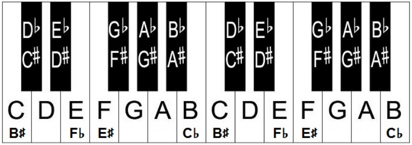
Some musical styles therefore avoid the major third in ET: these are the so-called power chords, where only the first and fifth are played (e.g. $C - G$, not $C - E - G$).

Advantages of ET: All keys are identical and we can use any key.

Disadvantages of ET: All keys are identical, but the price paid is that some intervals are unpleasant, such as the major third.
It's often said that ET is the most widely used temperament, which might be true for electronic keyboard instruments. It's less clear for pianos. My piano tuner does **not** use ET, although it's quite close, following measurement after tuning. He is really doing it by ear: his pitch memory is so good that he doesn't need electronic tuners.

Early Music: In the 20th century, many musicians became interested in reconstructing older instruments and playing old music in a historically considered way, i.e. not just playing it on modern instruments as if it were written now. This has led to much interest in temperaments which, combined with electronic tuning and digital keyboards which can change temperament easily, still continues.

The Pentatonic Scale: Do we need 12 notes in the scale? One alternative is just to use 5 notes separated by perfect fifths, with the root note then doubled in frequency to form the octave. It's easy to give an example: **the black notes**



Tuning is easy ($p = 3/2$):

$$F^\sharp \underset{p}{\rightarrow} C^\sharp \underset{p}{\rightarrow} G^\sharp \underset{p}{\rightarrow} D^\sharp \underset{p}{\rightarrow} A^\sharp$$

Many cultures have used this historically and it's still used: the fifth and the octave seem to be part of our auditory hardware. Pentatonic tunes: "Amazing Grace", "Auld Lang Syne", "Swing Low, Sweet Chariot", "Stairway to Heaven", ...

Pitch on stringed instruments: We can shorten the string (by pressing a finger against a fret) or increase the tension (by tightening a screw) or use a lower density string (compare strings for low and high notes).
16th century: Vincenzo Galilei (1520–1591) discovered that the pitch of a vibrating string is proportional to the square root of the tension in the string. Specifically, he found that weights suspended from identical strings of equal length needed to be in the ratio $9:4$ to produce the $3:2$ perfect fifth.
Mersenne's Laws (1636): In brief

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

Here, f is the frequency in Hertz (Hz), L is the length of the string (in metres), T is the tension (in Newtons), and μ is the linear density (mass per unit length, in kg per metre).

Example

The strings in an upright piano usually have a tension between 750N and 900N. Suppose our piano strings have a tension of 900N. The standard pitch for the A above middle C is defined to be 440Hz. If we use a string with linear density 0.0049kg/m, how long should the A string be?

Solution: We have $f = 440$ Hz, $T = 900$ N and $\mu = 0.0049\mathrm{kg/m}$, and we want to find L. We rearrange Mersenne's Law

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

to find

$$L = \frac{1}{2f}\sqrt{\frac{T}{\mu}} = \frac{1}{880}\sqrt{\frac{900}{0.0049}}$$
$$= \frac{30}{880 \times 0.07} \approx 0.48701299 \text{ m}.$$

Exercise

What is the vibrating frequency, in Hertz, of a string of length 50 *cm whose tension is* 100 *N, and whose linear density is* 0.16 *kg/m?*

Exercise

Suppose a string has a vibrating frequency of f.

1. *What is its new vibrating frequency* f_1 *if the tension in the string is halved but its length is also halved?*
2. *What is the difference in cents between* f *and* f_1*?*

ENTIRELY OPTIONAL temperament information:

12 true perfect fifths $\neq$ 7 octaves:

$$\mathrm{PC} = \frac{(3/2)^{12}}{2^7} = \frac{3^{12}}{2^{19}} \approx 1.01364.$$

PC is called the Pythagorean comma and

$$1200 \frac{\ln PC}{\ln 2} \approx 24 \text{ cents}.$$

One way to specify a temperament is to let the frequency ratios be

$$r_k = p \; PC^{q_k}, \quad \text{for} \quad 1 \leq k \leq 12,$$

where $p = 3/2$. Now

$$r_1 r_2 \cdots r_{12} = p^{12} PC^{q_1 + \cdots + q_{12}} = 2^7$$

or

$$PC^{q_1 + \cdots + q_{12}} = \frac{2^7}{p^{12}} = \mathrm{PC}^{-1}$$

i.e.

$$q_1 + q_2 + \cdots + q_{12} = -1.$$

OPTIONAL: Thus we can either use the frequency ratios r_k

$$A^\flat \underset{r_1}{\to} E^\flat \underset{r_2}{\to} B^\flat \underset{r_3}{\to} F \underset{r_4}{\to} C \underset{r_5}{\to} G \underset{r_6}{\to} \mathrm{D} \underset{r_7}{\to} A \underset{r_8}{\to} E \underset{r_9}{\to} B \underset{r_{10}}{\to} F^\sharp \underset{r_{11}}{\to} C^\sharp \underset{r_{12}}{\to} G^\sharp$$

or the q_k, where $r_k = p\,\mathrm{PC}^{q_k}$ and $r_1 r_2 \cdots r_{12} = 2^7$ or, equivalently,

$$q_1 + q_2 + \cdots + q_{12} = -1,$$

which gives the diagram

$$A^\flat \underset{q_1}{-} E^\flat \underset{q_2}{-} B^\flat \underset{q_3}{-} F \underset{q_4}{-} C \underset{q_5}{-} G \underset{q_6}{-} \mathrm{D} \underset{q_7}{-} A \underset{q_8}{-} E \underset{q_9}{-} B \underset{q_{10}}{-} F^\sharp \underset{q_{11}}{-} C^\sharp \underset{q_{12}}{-} G^\sharp,$$

What's the point? We often start by tuning a pure fifth, because that's easy to hear, and then modify it. It's less necessary given modern electronic tuners, but it's still useful.

OPTIONAL: Recall Vallotti:

$$A^\flat \underset{p}{\to} E^\flat \underset{p}{\to} B^\flat \underset{p}{\to} F \underset{\beta}{\to} C \underset{\beta}{\to} G \underset{\beta}{\to} \mathrm{D} \underset{\beta}{\to} A \underset{\beta}{\to} E \underset{\beta}{\to} B \underset{p}{\to} F^\sharp \underset{p}{\to} C^\sharp \underset{p}{\to} G^\sharp,$$

where $p = 3/2$ and $p^6\beta^6 = 2^7$. If $\beta = p\ PC^\gamma$, then $6\gamma = -1$, i.e. $\gamma = -1/6$, and displaying the q_k only we find

$$A^\flat \underset{0}{-} E^\flat \underset{0}{-} B^\flat \underset{0}{-} F \underset{-\frac{1}{6}}{-} C \underset{-\frac{1}{6}}{-} G \underset{-\frac{1}{6}}{-} \mathrm{D} \underset{-\frac{1}{6}}{-} A \underset{-\frac{1}{6}}{-} E \underset{-\frac{1}{6}}{-} B \underset{0}{-} F^\sharp \underset{0}{-} C^\sharp \underset{0}{-} G^\sharp,$$

PC $\approx$ 24 cents, so $-1/6$ means it's roughly $24/6 = 4$ cents flat.

Exercise

Use this notation to display Young II and Barnes.

OPTIONAL: Werckmeister III: Recall that

$$A^\flat \underset{p}{\to} E^\flat \underset{p}{\to} B^\flat \underset{p}{\to} F \underset{p}{\to} C \underset{\alpha}{\to} G \underset{\alpha}{\to} \mathrm{D} \underset{\alpha}{\to} A \underset{p}{\to} E \underset{p}{\to} B \underset{\alpha}{\to} F^\sharp \underset{p}{\to} C^\sharp \underset{p}{\to} G^\sharp,$$

where $p = 3/2$ and $p^8\alpha^4 = 2^7$.

Exercise

Show that $\alpha = 2^{7/4}/p^2$. *If* $\alpha = pA$, *show that* $A = \mathrm{PC}^{-1/4}$.

Hence Werckmeister III corresponds to

$$A^\flat \underset{0}{-} E^\flat \underset{0}{-} B^\flat \underset{0}{-} F \underset{0}{-} C \underset{-\frac{1}{4}}{-} G \underset{-\frac{1}{4}}{-} \mathrm{D} \underset{-\frac{1}{4}}{-} A \underset{0}{-} E \underset{0}{-} B \underset{-\frac{1}{4}}{-} F^\sharp \underset{0}{-} C^\sharp \underset{0}{-} G^\sharp,$$

PC $\approx$ 24 cents, so $-1/4$ means it's roughly $24/4 = 6$ cents flat.

OPTIONAL: For ET, every ratio $r_k = 2^{7/12}$. Thus we have $r_k = p\ \mathrm{PC}^\gamma$ and $12\gamma = -1$, or $\gamma = -1/12$.

$$A^\flat \underset{\frac{-1}{12}}{-} E^\flat \underset{\frac{-1}{12}}{-} B^\flat \underset{\frac{-1}{12}}{-} F \underset{\frac{-1}{12}}{-} C \underset{\frac{-1}{12}}{-} G \underset{\frac{-1}{12}}{-} \mathrm{D} \underset{\frac{-1}{12}}{-} A \underset{\frac{-1}{12}}{-} E \underset{\frac{-1}{12}}{-} B \underset{\frac{-1}{12}}{-} F^\sharp \underset{\frac{-1}{12}}{-} C^\sharp \underset{\frac{-1}{12}}{-} G^\sharp$$

PC $\approx$ 24 cents, so $-1/12$ means it's roughly $24/12 = 2$ cents flat.

OPTIONAL: A new well-tempering: Bach/Lehman 2005:

$$A^\flat \underset{\frac{-1}{12}}{-} E^\flat \underset{\frac{-1}{12}}{-} B^\flat \underset{\frac{1}{12}}{-} F \underset{\frac{-1}{6}}{-} C \underset{\frac{-1}{6}}{-} G \underset{\frac{-1}{6}}{-} \mathrm{D} \underset{\frac{-1}{6}}{-} A \underset{\frac{-1}{6}}{-} E \underset{0}{-} B \underset{0}{-} F^\sharp \underset{0}{-} C^\sharp \underset{\frac{-1}{12}}{-} G^\sharp.$$

This is quite popular in modern performance, although Lehman's claim that this was Bach's temperament is disputed.

OPTIONAL: Lehman claims the curls encode his temperament: