1.

Need NOT always Taylor expand directly!

FOURIER TRANSFORM IN 1D

If

$$g(x) = f(x+h) - 2f(x) + f(x-h)$$

then

$$\hat{g}(z) = \left(e^{ihz} - 2 + e^{-ihz}\right)\hat{f}(z)$$

$$= \underbrace{2\left(\cos(hz) - 1\right)}\hat{f}(z)$$

where

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(x)\, e^{-ixz}\, dx.$$

So

$$\hat{g}(z) = 2\left(-\tfrac{1}{2}h^2z^2 + \tfrac{1}{24}h^4z^4 - \cdots\right)\hat{f}(z)$$

$$= -h^2z^2\hat{f}(z) + \tfrac{1}{12}h^4z^4\hat{f}(z) - \cdots.$$

Hence

$$g(x) = -h^2 f''(x) + \tfrac{1}{12}h^4 f^{(4)}(x) - \cdots.$$

FOURIER TRANSFORM IN 2D

Here $\hat{g}(z,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, e^{-i(xz+yw)}\, dx\,dy$

and $\widehat{\partial_{xx} g} = -z^2\hat{g}$, $\widehat{\partial_{yy} g} = -w^2\hat{g}$.

# 5-POINT DIFFERENCE EXAMPLE

Let

$$Lu(x,y) = u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y).$$

Then

$$\widehat{Lu}(z,w) = \underbrace{\left(e^{ihz} + e^{-ihz} + e^{ihw} + e^{-ihw} - 4\right)}_{\hat{L}}\hat{u}.$$

Hence

$$\begin{aligned}\hat{L} &= 2\cos(hz) + 2\cos(hw) - 4 \\ &= 2\Big(-\tfrac{1}{2}h^2z^2 + \tfrac{1}{24}h^4z^4 - \cdots \\ &\qquad -\tfrac{1}{2}h^2w^2 + \tfrac{1}{24}h^4w^4 - \cdots\Big) \\ &= -h^2(z^2+w^2) + O(h^4).\end{aligned}$$

So

$$Lu(x,y) = -h^2\nabla^2 u(x,y) + O(h^4).$$

Now use the FT to complete Qu. 1.

Here $\quad Lu = h^2 \nabla^2 u \quad$ where

$$\hat{L} = -\frac{10}{3} + \frac{2}{3}\left(e^{ihz} + e^{-ihz} + e^{ihw} + e^{-ihw}\right)$$
$$+ \frac{1}{6}\left(e^{ihz} + e^{-ihz}\right)\left(e^{ihw} + e^{-ihw}\right)$$

$$= -\frac{10}{3} + \frac{4}{3}\left(\cos hz + \cos hw\right)$$
$$+ \frac{2}{3} \cos hz \cos hw .$$

Now we Taylor expand the cosines:

$$\hat{L} = -\frac{10}{3} + \frac{4}{3}\Big(2 - \frac{1}{2}h^2z^2 + \frac{1}{24}h^4z^4 - \cdots$$
$$- \frac{1}{2}h^2w^2 + \frac{1}{24}h^4w^4 - \cdots\Big)$$
$$+ \frac{2}{3}\left(1 - \frac{1}{2}h^2z^2 + \frac{1}{24}h^4z^4 - \cdots\right)$$
$$\times\left(1 - \frac{1}{2}h^2w^2 + \frac{1}{24}h^4w^4 - \cdots\right)$$

$O(1)$: $\quad -\frac{10}{3} + \frac{8}{3} + \frac{2}{3} = 0.$

$O(h^2)$: $\quad -\frac{2}{3}z^2 - \frac{2}{3}w^2 - \frac{1}{3}z^2 - \frac{1}{3}w^2$
$$= -(z^2 + w^2).$$

$O(h^4)$: $\quad \frac{4}{3} \cdot \frac{1}{4!}(z^4 + w^4)$

$$+ \frac{2}{3} \cdot \frac{1}{4!} (z^4 + w^4) - \frac{1}{6} z^2 w^2$$

$$= \frac{1}{12} (z^4 + w^4 - 2z^2 w^2)$$

$$= \frac{1}{12} (z^2 + w^2)^2 .$$

Hence

$$Lu = h^2 \nabla^2 u + \frac{1}{12} h^4 \nabla^4 u + O(h^6)$$

$$= \begin{cases} O(h^4) & \text{general } u \\ O(h^6) & \text{if } \nabla^2 u = 0 . \end{cases}$$

OLDER METHOD: OPERATOR CALCULUS

$$\hat{L} = -\frac{10}{3} + \frac{2}{3}\left(e^{h\partial_x} + e^{-h\partial_x} + e^{h\partial_y} + e^{-h\partial_y}\right)$$
$$+ \frac{1}{6}\left(e^{h\partial_x} + e^{-h\partial_x}\right)\left(e^{h\partial_y} - e^{-h\partial_y}\right)$$

$$= -\frac{10}{3} + \frac{4}{3}\left(\cosh(h\partial_x) + \cosh(h\partial_y)\right)$$
$$+ \frac{2}{3} \cosh(h\partial_x) \cosh(h\partial_y)$$

$$= -\frac{10}{3} + \frac{4}{3}\left(2 + \frac{1}{2} h^2 \partial_x^2 + \frac{1}{24} h^4 \partial_x^4 + \frac{1}{2} h^2 \partial_y^2 + \frac{1}{24} h^4 \partial_y^4 + O(h^6)\right)$$
$$+ \frac{2}{3}\left(1 + \frac{1}{2} h^2 \partial_x^2 + \frac{1}{24} h^4 \partial_x^4 + O(h^6)\right)$$
$$\times \left(1 + \frac{1}{2} h^2 \partial_y^2 + \frac{1}{24} h^4 \partial_y^4 + O(h^6)\right)$$

$$O(1) = -\frac{10}{3} + \frac{8}{3} + \frac{2}{3} = 0$$

$$O(h^2) = \frac{2}{3}(\partial_x^2 + \partial_y^2) + \frac{1}{3}(\partial_x^2 + \partial_y^2) = \nabla^2$$

$$O(h^4) = \frac{4}{3}\cdot\frac{1}{24}(\partial_x^4 + \partial_y^4) + \frac{2}{3}\cdot\frac{1}{24}(\partial_x^4 + \partial_y^4)$$
$$+ \frac{1}{6}\partial_x^2\partial_y^2$$
$$= \frac{1}{12}(\partial_x^4 + \partial_y^4 + 2\partial_x^2\partial_y^2)$$
$$= \frac{1}{12}\nabla^2(\nabla^2).$$

So

$$h^{-2}L\,u(x,y) = \nabla^2 u(x,y) + \frac{1}{12}h^2\nabla^2(\nabla^2 u) + O(h^4).$$

EXAMPLE: If $g(x) = h^{-2}\big(f(x+h) - 2f(x) + f(x-h)\big)$

then
$$f(x+h) = \underbrace{\sum_{\ell=0}^{\infty}\frac{h^\ell D^\ell}{\ell!}}_{e^{hD}} f(x) \qquad \left(D \equiv \frac{d}{dx}\right)$$

(SEE FOURIER METHOD FOR RIGOR)

Hence
$$g(x) = h^{-2}\left(e^{hD} - 2 + e^{-2hD}\right) f(x)$$
$$= 2h^{-2}\left(\cosh(hD) - 1\right) f(x)$$
$$= D^2 f(x) + \frac{1}{12}h^2 D^4 f(x) + O(h^4).$$

[This is Fourier transform in disguise: $D = iz$]