Do NOT ALWAYS Taylor expand directly!

FOURIER TRANSFORM IN 1D

If
$$g(x) = f(x+h) - 2f(x) + f(x-h)$$
then
$$\hat{g}(z) = (e^{ihz} - 2 + e^{-ihz})\hat{f}(z)$$
$$= 2(\cos(hz) - 1)\hat{f}(z)$$
where
$$\hat{f}(z) = \int_{-\infty}^{\infty} f(x)\, e^{-ixz}\, dx.$$

So
$$\hat{g}(z) = 2\left(-\tfrac{1}{2}h^2z^2 + \tfrac{1}{24}h^4z^4 - \cdots\right)\hat{f}(z)$$
$$= -h^2z^2\hat{f}(z) + \tfrac{1}{12}h^4z^4\hat{f}(z) - \cdots$$
Hence
$$g(x) = -h^2 f''(x) + \tfrac{1}{12}h^4 f^{(4)}(x) - \cdots$$

FOURIER TRANSFORM IN 2D

Here
$$\hat{g}(z,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, e^{-i(xz+yw)}\, dx\,dy$$

and
$$\widehat{\partial_x g} = -z^2\hat{g}, \quad \widehat{\partial_y g} = -w^2\hat{g}.$$

5-POINT DIFFERENCE EXAMPLE

Let

$$Lu(x,y) = u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y).$$

Then

$$\widehat{Lu}(z,w) = \underbrace{\left(e^{ihz} + e^{-ihz} + e^{ihw} + e^{-ihw} - 4\right)}_{\hat{L}}\hat{u}$$

Hence

$$\hat{L} = 2\cos(hz) + 2\cos(hw) - 4$$

$$= 2\left(-\tfrac{1}{2}h^2z^2 + \tfrac{1}{24}h^4z^4 - \cdots - \tfrac{1}{2}h^2w^2 + \tfrac{1}{24}h^4w^4 - \cdots\right)$$

$$= -h^2(z^2+w^2) + O(h^4).$$

So

$$Lu(x,y) = -h^2\nabla^2 u(x,y) + O(h^4).$$

NOW USE THE FT to complete Qn. 1.