25. NOTATION: Use $\underline{u}^k$ instead of $\underline{u}^{(k)}$ to avoid brackets!

$$(1) \quad \begin{cases} u_{m+1} - 2u_m + u_{m-1} = 2h^2, & m = 1:M, \quad \left(h = \frac{1}{M}\right) \\ u_0 = u_M = 0. \end{cases}$$

TST theory: The eigenvalues of H are

$$\lambda_p = \cos\frac{\pi p}{M}, \qquad p = 1:M-1,$$

with corresponding eigenvectors $\underline{v}^1, \ldots, \underline{v}^{M-1}$, where

$$v_m^p = \sin\frac{\pi p m}{M}, \qquad m = 1:M-1.$$

CLAIM: $\rho(H) = \cos\frac{\pi}{M} = \lambda_1$.

Proof: λ_p is a decreasing sequence since $\cos \pi x$ is decreasing for $0 \le x \le 1$, so $\rho(H) = \max\{|\lambda_1|, |\lambda_{M-1}|\}$. Finally, $\lambda_{M-1} = \cos\frac{\pi(M-1)}{M} = -\cos\frac{\pi}{M} = -\lambda_1$, so $\rho(H) = \lambda_1$. $\square$

Now $(\underline{v}^p)^T(\underline{v}^q) = \frac{M}{2}\cdot\delta_{pq}$ (EXERCISE).

INNER PRODUCT: Define

$$\langle \underline{a}, \underline{b} \rangle = \frac{2}{M}\sum_{\ell=1}^{M-1} a_\ell b_\ell, \qquad \underline{a}, \underline{b} \in \mathbb{R}^{M-1}.$$

Then $\underline{v}^1, \ldots, \underline{v}^{M-1}$ are orthonormal and

$$\underline{e}^0 = \sum_{p=1}^{M-1} \langle \underline{e}^0, \underline{v}^p \rangle \underline{v}^p$$

while $\underline{e}^k = H^k \underline{e}^0 = \sum_{p=1}^{M-1} \lambda_p^k \langle \underline{e}^0, \underline{v}^p \rangle \underline{v}^p$.

HIGH FREQUENCIES DAMPED FASTER: $\underline{e}^k \simeq \lambda_1^k \langle \underline{e}^0, \underline{v}^1 \rangle \underline{v}^1$.

Thus largest component $\simeq \lambda_1^k \langle \underline{e}^0, \underline{v}^1 \rangle = \cos^k\frac{\pi}{M} \cdot \langle \underline{e}^0, \underline{v}^1 \rangle$

INTEGRAL APPROXIMATION:

$$\begin{aligned}
\langle \underline{e}^0, \underline{v}^1 \rangle &= -\langle \underline{u}^\infty, \underline{v}^1 \rangle \\
&= -\frac{2}{M}\sum_{q=1}^{M-1} u\left(\frac{q}{M}\right)\sin\frac{\pi q}{M} \\
&\underset{(\text{BIG } M)}{\simeq} -2\int_0^1 u(s)\sin \pi s \, ds.
\end{aligned}$$

EXERCISE: $-2\int_0^1 (s^2 - s)\sin \pi s \, ds = \frac{8}{\pi^3}$.

Thus largest component $\simeq \frac{8}{\pi^3} \cos^k \frac{\pi}{M}$.

ITERATION ESTIMATE: Want k s.t.

$$\frac{8}{\pi^3} \cos^k \frac{\pi}{M} \leq \delta . \qquad (2)$$

Now $\cos \frac{\pi}{M} = 1 - \frac{\pi^2}{2M^2} + \cdots \simeq e^{-\frac{\pi^2}{2M^2}}$, so (2) is

$$\frac{8}{\pi^3} e^{-\frac{k\pi^2}{2M^2}} \leq \delta$$

or $\log \frac{8}{\pi^3} - \frac{k\pi^2}{2M^2} \leq \log \delta$, $\qquad (3)$

i.e

$$k \geq \frac{2M^2}{\pi^2} \log\left(\frac{8}{\pi^3 \delta}\right) . \qquad (4)$$

Crucial part here is $k \geq O(M^2)$, i.e. SLOW.

Finally, $\frac{2}{\pi^2} \log\left(\frac{8}{\pi^3 10^{-6}}\right) \simeq 2.53$.