

Real Analysis 1: Foundations

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You can download these slides and the lecture videos from my office server

<http://econ109.econ.bbk.ac.uk/brad/analysis/>

RECOMMENDED BOOK: "A First Course in
Mathematical Analysis"
by J. C. BURKILL
[Cambridge University Press]

Analysis is calculus made rigorous

1600s–1700s: Calculus created by Newton, Leibniz and others.

1700s–1800s: Massive expansion of calculus drove progress in science and engineering, but found **worrying problems**. Centuries of handwaving had led to powerful results, but only partially understood.

1850– early 1900s: Calculus and the foundations of mathematics made rigorous. Here are the bare bones (and much is omitted):

- Infinite sets (Cantor); construction of the real numbers (Cantor, Dedekind, Weierstrass)
- Convergence of sequences and series (Cauchy, Weierstrass)
- Continuity and differentiability; convergence of Taylor series (Cauchy, Weierstrass)
- Integration (Riemann)

Analysis didn't end in the 19th century. This is where the mathematics of this course led in the 20th century:

- Early 1900s: Integration again (Borel and Lebesgue) – Riemann integration wasn't enough.
- 1920s–1940s: Foundations of Probability Theory (Kolmogorov, Steinhaus and Wiener) led to new concepts, e.g. Brownian motion, and a new understanding of probability and statistics.
- 1900s–1940s: Mathematical foundations (Peano, Russell, Hilbert, Turing) and the precise concept of an algorithm lead to new theoretical insights (Gödel's incompleteness theorem) and, ultimately, the computer and programming languages.
- 1930s–: New algebraic structures (Hilbert, Banach, Steinhaus) unify analytical concepts: Banach and Hilbert spaces.

The natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$.

The integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

The rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

BUT there is no rational whose square is 2.

We can construct sequences of rationals that approximate $\sqrt{2}$ as closely as we wish. We shall see later that, if we let $a_0 = 1$ and define

$$a_n = \frac{1}{2} \left(a_{n-1} + \frac{2}{a_{n-1}} \right), \quad \text{for } n \in \mathbb{N},$$

then (a_n) is a sequence of rationals for which $a_n^2 \rightarrow 2$, but the crucial point is that we require **infinitely** many arithmetic operations to do this.

Modern decimal notation (John Napier, early 1600s) makes this look deceptively easy:

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694 \dots$$

but this is really an infinite series, i.e.

$$\sqrt{2} = 1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \dots$$

How do we prove that it converges **and** that the algebraic properties of \mathbb{Q} extend to the new system? It's the extension requiring infinitely many operations that divides analysis from algebra.

\mathbb{Q} is a **field**:

Definition: A **field** is a set \mathbb{F} together with commutative binary operations $+$ and \times satisfying the following axioms:

- 1 For all $a, b, c \in \mathbb{F}$, $a \times (b + c) = (a \times b) + (a \times c)$.
- 2 $+$, \times have identity elements $0_{\mathbb{F}}$, $1_{\mathbb{F}}$, respectively: for all $a \in \mathbb{F}$, $a + 0_{\mathbb{F}} = a$ and $a \times 1_{\mathbb{F}} = a$.
- 3 For all $a \in \mathbb{F}$, there exists $-a \in \mathbb{F}$ such that $a + (-a) = 0_{\mathbb{F}}$.
- 4 For all $a \in \mathbb{F} \setminus \{0_{\mathbb{F}}\}$, there exists $a^{-1} \in \mathbb{F}$ such that $a \times a^{-1} = 1_{\mathbb{F}}$.

(\quad)

\mathbb{Q} has more structure: \mathbb{Q} is an **ordered field**:

Definition An **ordered field** is a field \mathbb{F} with a subset $\mathbb{F}_+ \subset \mathbb{F}$ satisfying

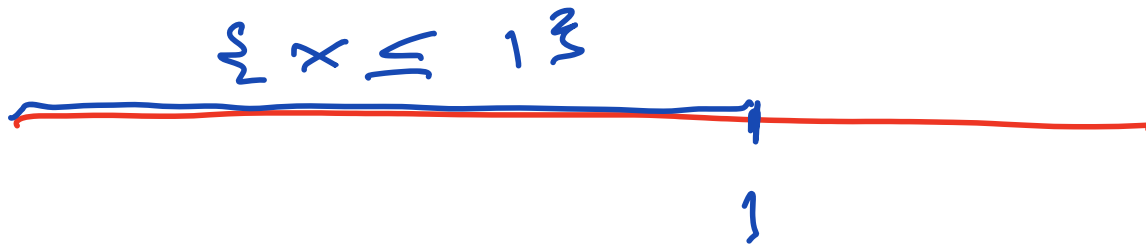
- 1 For all $a, b \in \mathbb{F}_+$, $a + b \in \mathbb{F}_+$ and $a \times b \in \mathbb{F}_+$.
- 2 For all $a \in \mathbb{F}$, exactly one of the following is true: **either** $a \in \mathbb{F}_+$ **or** $a = 0_{\mathbb{F}}$ **or** $-a \in \mathbb{F}_+$.

We say that \mathbb{F}_+ is the set of **positive elements** of \mathbb{F} and we define

$$a <_{\mathbb{F}} b \quad \text{if and only if} \quad b - a \in \mathbb{F}_+.$$

\mathbb{Q} is an ordered field. We shall construct \mathbb{R} as an extension of \mathbb{Q} that is a **complete ordered field**, i.e. it contains all the irrationals, such as $\sqrt{2}$. To use the terminology we establish later in the course, \mathbb{R} is essentially \mathbb{Q} augmented by all convergent sequences of rationals. But we need some definitions first.

Definition Given any subset $A \subset \mathbb{R}$, A is **bounded above** if there exists $u \in \mathbb{R}$ for which $a \leq u$ for all $a \in A$. If $U \in \mathbb{R}$ is an upper bound for which $U \leq u$ for **every** upper bound u of A , then we say that U is the **least upper bound**. There are two common notations for this: $\text{lub } A$ and $\text{sup } A$.



SUP = SUPREMUM
(LATIN: BIGGEST)

If l_1, l_2 are 2 lubs, then $l_1 \leq l_2$ and $l_2 \leq l_1$, so $l_1 = l_2$.

Exercise: Prove that there is exactly one least upper bound, i.e. it's unique.

Similarly, we say A is **bounded below** if there exists $\ell \in \mathbb{R}$ for which $\ell \leq a$ for all $a \in A$. If $L \in \mathbb{R}$ is the greatest lower bound for A , then we write $L = \inf A$ or $L = \text{glb } A$.

Exercise: Let

$$A = \{0, 1/2, 2/3, 3/4, 4/5, \dots\}.$$

Show that $\inf A = 0$ and $\sup A = 1$.

INF = INFIMUM

(LATIN: SMALLEST)

$$\frac{n}{n+1} = 1 - \frac{1}{n+1}$$

In the language we shall explore later in this course, \mathbb{R} is \mathbb{Q} extended by including all convergent sequences of rationals. We can avoid sequences for the moment in a very neat way:

The Completeness Axiom: Every non-empty subset of \mathbb{R} that is bounded above has exactly one **least upper bound** in \mathbb{R} .

Example: If $A = \{x \in \mathbb{R} : x^2 < 2\}$, then $a = \sup A \in \mathbb{R}$ satisfies $a^2 = 2$.

Suppose $a^2 < 2$. Now

$$\left(a + \frac{1}{n}\right)^2 = a^2 + \frac{2a}{n} + \frac{1}{n^2} \leq a^2 + \frac{2a + 1}{n}.$$

We know that there exists $n_0 \in \mathbb{N}$ such that

$$n_0 > \frac{2a + 1}{2 - a^2},$$

for otherwise \mathbb{N} would be bounded. Hence

$$\left(a + \frac{1}{n_0}\right)^2 \leq a^2 + \frac{2a + 1}{n_0} < a^2 + 2 - a^2 = 2.$$

But then $a + \frac{1}{n_0} < a$ which is nonsense.

Exercise Show that $a^2 > 2$ also leads to a contradiction.

Crucial Fact: The Completeness Axiom is **FALSE** in \mathbb{Q} .

Thus there is no rational number r which satisfies

$$r = \sup\{q \in \mathbb{Q} : q^2 < 2\}.$$

Theorem (Approximation Property)

Let S be a nonempty subset of \mathbb{R} and let $U = \sup S$. Then, for every $a < U$, there exists $x \in S$ for which $a < x \leq U$.

\emptyset $\frac{1}{2}$ \cdot \setminus



Proof.

If we had $x \leq a$ for every $x \in S$, then a would be a smaller upper bound than $U = \sup S$, contradicting the definition of $\sup S$.

Therefore $x > a$ for at least one $x \in S$. □

Theorem

\mathbb{N} is unbounded above.

Proof.

If \mathbb{N} were bounded above, then $U = \sup \mathbb{N} \in \mathbb{R}$, by the Completeness Axiom. By the Approximation Property, there would exist some $n \in \mathbb{N}$ for which $U - 1 < n$. But then $n + 1 > U$, i.e. U is **not** an upper bound, which is a contradiction. □

Theorem

Let $x \in \mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $n > x$.

Proof.

If this were not true, then \mathbb{N} would be bounded above. □

Theorem (The Archimedean Property of \mathbb{R})

If $x > 0$ and $y \in \mathbb{R}$, then there is a positive integer n for which $nx > y$.

Proof.

There is a positive integer n exceeding y/x . □

It's time to return to some actual numbers. A real number of the form

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n},$$

where a_0 is a non-negative integer and a_1, \dots, a_n are integers satisfying $0 \leq a_k \leq 9$ is usually written as

$$r = a_0.a_1a_2 \cdots a_n.$$

This is called a **finite decimal representation** of r .

Theorem (Arbitrarily accurate decimal approximations exist.)

Let $x \in \mathbb{R}_+$. Then for every integer $n \geq 1$ there exists a finite decimal $r_n = a_0.a_1a_2 \cdots a_n$ such that

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

This is really our first algorithm, i.e. essentially a computer program: we call this a **constructive proof**.

Proof.

Let

$$* S = \{n \in \mathbb{N} : 0 \leq n \leq x\}.$$

Then $a_0 = \sup S$ is a non-negative integer and we write $a_0 = [x]$, the greatest integer $\leq x$. Thus

$$a_0 \leq x < a_0 + 1.$$

$$[3.1] = 3$$

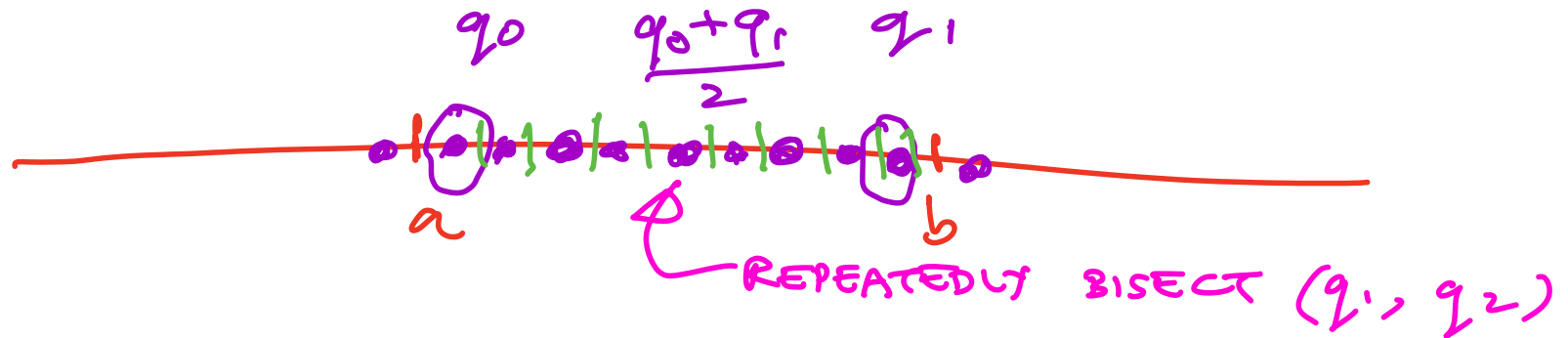
$$[3.9] = 3$$

Now let $a_1 = [10x - 10a_0]$, i.e. the greatest integer $\leq 10x - 10a_0$. We have $0 \leq 10x - 10a_0 = 10(x - a_0) < 10$, so $0 \leq a_1 \leq 9$ and

$$a_1 \leq 10x - 10a_0 < a_1 + 1, \quad \text{or} \quad a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1 + 1}{10}.$$

The algorithm then continues with $a_2 = [10^2x - 10^2a_0 - 10a_1]$.





The previous result tells us two important facts:

- ① We can find infinitely many rational numbers between any two real numbers.
- ② We can approximate any real number as closely as we wish.

We say that \mathbb{Q} is **dense** in \mathbb{R} .

Example: Between any two real numbers there is an irrational number.

To see this, let the interval be (a, b) . The trick is to consider the interval $(a + \sqrt{2}, b + \sqrt{2})$. We know that this interval contains a rational number, q say, i.e.

$$a + \sqrt{2} < q < b + \sqrt{2}.$$

Hence

$$a < q - \sqrt{2} < b.$$

Exercise: Prove that $q - \sqrt{2}$ is irrational if $q \in \mathbb{Q}$.

Exercise: Prove that there are infinitely many irrational numbers in (a, b) .

\downarrow
 $q - \sqrt{2} = \frac{m}{n}$
then
 $\sqrt{2} = q - \frac{m}{n}$
 $\in \mathbb{Q}$ ~~\mathbb{Q}~~

One of the motivations for analysis was Cantor's new way of comparing infinite sets.

Definition We say that any two sets X and Y have the same **cardinality** if there is a bijection $f : X \rightarrow Y$.

Thus two sets have the same cardinality if their elements can be paired up. In particular, we say that a set X has cardinality $n \in \mathbb{N}$ if there is a bijection $f : X \rightarrow \{1, 2, \dots, n\}$.

Cantor's insight was that we could use this definition to compare infinite sets too. For example, if we let $2\mathbb{Z}$ denote the even integers, i.e.

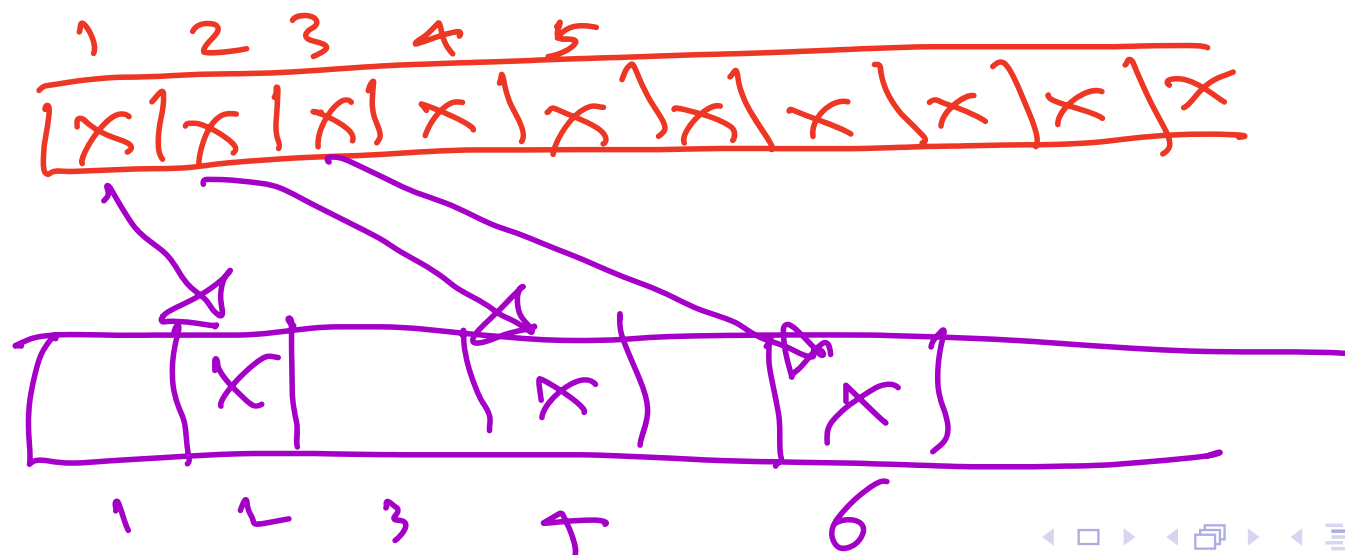
$$2\mathbb{Z} = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\},$$

then $f : \mathbb{Z} \rightarrow 2\mathbb{Z}$ defined by $f(n) = 2n$, $n \in \mathbb{Z}$, is a bijection. Thus the even integers have the same cardinality as the integers, even though $2\mathbb{Z}$ is a proper subset of \mathbb{Z} .

Definition We say that a set X is **countable** if it's a finite set or if it has the same cardinality as the integers.

Exercise: Show that the set of all odd integers is countable.

HILBERT'S HOTEL : COUNTABLY INFINITE ROOMS,
COUNTABLY MANY NEW GUESTS



Theorem

The set

$$\mathbb{N} \times \mathbb{N} = \{(j, k) : j, k \in \mathbb{N}\}$$

is countable.

Proof.

Define $f(j, k) = 2^j 3^k$. Then f is injective (Why?) from $\mathbb{N} \times \mathbb{N}$ into a subset of \mathbb{N} . □

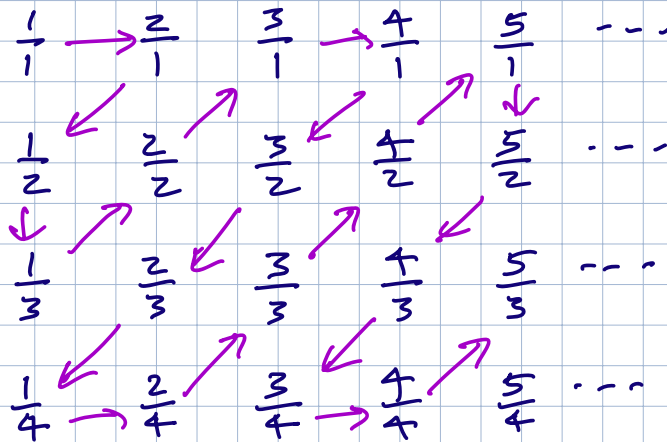
Theorem

\mathbb{Q} *is countable.*

Proof.

The function $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}$ given by $g(m, n) = m/n$ is sufficient, but I will give further details in the lecture. □

\mathbb{Q} IS COUNTABLE



Row COL
Box (j, k)
is $\frac{k}{j}$

PURPLE PATH ENUMERATES \mathbb{Q} (MULTIPLE TIMES),

SO \mathbb{Q} IS COUNTABLE.

Theorem

$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is not countable.

Cantor's diagonal argument sketch.

If $(0, 1)$ were countable, then it would be actually be a sequence a_1, a_2, \dots , where each a_n is an infinite decimal

$$a_n = 0.a_{n,1}a_{n,2} \dots, \quad \text{for } n \in \mathbb{N}.$$

Now define a new $A = 0.A_1A_2 \dots \in \mathbb{R}$ by

$$A_n = \begin{cases} 1 & \text{if } a_{n,n} \neq 1, \\ 2 & \text{if } a_{n,n} = 1. \end{cases}$$

Then A differs from a_n in the n^{th} decimal place, a contradiction. □

Nonexaminable fun: almost all reals are irrational.

To see this, we shall sketch some ideas from more advanced analysis. Let q_1, q_2, \dots denote the rational numbers in $(0, 1)$.

Given any $\epsilon > 0$, let

$$I_n = \left(q_n - \frac{\epsilon}{2^{n+1}}, q_n + \frac{\epsilon}{2^{n+1}} \right), \quad \text{for } n \in \mathbb{N}.$$



Then $q_n \in I_n$ and the length of I_n is $L_n = \frac{\epsilon}{2^n}$. Thus $\mathbb{Q} \cap (0, 1)$ is contained in $I_1 \cup I_2 \cup \dots$ and the “length” of $\mathbb{Q} \cap (0, 1)$ should be less than

$$L_1 + L_2 + L_3 + \dots = \epsilon \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \right) = \epsilon.$$

Since $\epsilon > 0$ can be as small as we wish, we say that $\mathbb{Q} \cap (0, 1)$ has **measure zero**, while $(0, 1)$ has measure 1.