Real Analysis 1: Foundations

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May 1, 2023

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`http://econ109.econ.bbk.ac.uk/brad/analysis/`

Analysis is calculus made rigorous

1600s–1700s: Calculus created by Newton, Leibniz and others.

1700s–1800s: Massive expansion of calculus drove progress in science and engineering, but found worrying problems. Centuries of handwaving had led to powerful results, but only partially understood.

1850– early 1900s: Calculus and the foundations of mathematics made rigorous. Here are the bare bones (and much is omitted):

- Infinite sets (Cantor); construction of the real numbers (Cantor, Dedekind, Weierstrass)
- Convergence of sequences and series (Cauchy, Weierstrass)
- Continuity and differentiability; convergence of Taylor series (Cauchy, Weierstrass)
- Integration (Riemann)

Analysis didn't end in the 19th century. This is where the mathematics of this course led in the 20th century:

- Early 1900s: Integration again (Borel and Lebesgue) – Riemann integration wasn't enough.
- 1920s–1940s: Foundations of Probability Theory (Kolmogorov, Steinhaus and Wiener) led to new concepts, e.g. Brownian motion, and a new understanding of probability and statistics.
- 1900s–1940s: Mathematical foundations (Peano, Russell, Hilbert, Turing) and the precise concept of an algorithm lead to new theoretical insights (Gödel's incompleteness theorem) and, ultimately, the computer and programming languages.
- 1930s–: New algebraic structures (Hilbert, Banach, Steinhaus) unify analytical concepts: Banach and Hilbert spaces.

The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.
The integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
The rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

BUT there is no rational whose square is 2.
We can construct sequences of rationals that approximate $\sqrt{2}$ as closely as we wish. We shall see later that, if we let $a_0 = 1$ and define

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}} \right), \qquad \text{for } n \in \mathbb{N},$$

then (a_n) is a sequence of rationals for which $a_n^2 \to 2$, but the crucial point is that we require **infinitely** many arithmetic operations to do this.

Modern decimal notation (John Napier, early 1600s) makes this look deceptively easy:

$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769 4 \cdots$

but this is really an infinite series, i.e.

$$\sqrt{2} = 1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \cdots$$

How do we prove that it converges **and** that the algebraic properties of $\mathbb{Q}$ extend to the new system? It's the extension requiring infinitely many operations that divides analysis from algebra.

$\mathbb{Q}$ is a **field**:

Definition

A **field** *is a set* $\mathbb{F}$ *together with commutative binary operations* $+$ *and* $\times$ *satisfying the following axioms:*

1. *For all* $a, b, c \in \mathbb{F}$, $a \times (b + c) = (a \times b) + (a \times c)$.
2. $+$, $\times$ *have identity elements* $0_{\mathbb{F}}$, $1_{\mathbb{F}}$, *respectively: for all* $a \in \mathbb{F}$, $a + 0_{\mathbb{F}} = a$ *and* $a \times 1_{\mathbb{F}} = a$.
3. *For all* $a \in \mathbb{F}$, *there exists* $-a \in \mathbb{F}$ *such that* $a + (-a) = 0_{\mathbb{F}}$.
4. *For all* $a \in \mathbb{F} \setminus \{0_{\mathbb{F}}\}$, *there exists* $a^{-1} \in \mathbb{F}$ *such that* $a \times a^{-1} = 1_{\mathbb{F}}$.

$\mathbb{Q}$ has more structure: $\mathbb{Q}$ is an **ordered field**:

Definition

An **ordered field** *is a field $\mathbb{F}$ with a subset $\mathbb{F}_+ \subset \mathbb{F}$ satisfying*

1. *For all $a, b \in \mathbb{F}_+$, $a + b \in \mathbb{F}_+$ and $a \times b \in \mathbb{F}_+$.*
2. *For all $a \in \mathbb{F}$, exactly one of the following is true:* **either** $a \in \mathbb{F}_+$ **or** $a = 0_{\mathbb{F}}$ **or** $-a \in \mathbb{F}_+$.

We say that $\mathbb{F}_+$ is the set of **positive elements** of $\mathbb{F}$ and we define

$$a <_{\mathbb{F}} b \quad \text{if and only if} \quad b - a \in \mathbb{F}_+.$$

$\mathbb{Q}$ is an ordered field. We shall construct $\mathbb{R}$ as an extension of $\mathbb{Q}$ that is a **complete ordered field**, i.e. it contains all the irrationals, such as $\sqrt{2}$. To use the terminology we establish later in the course, $\mathbb{R}$ is essentially $\mathbb{Q}$ augmented by all convergent sequences of rationals. But we need some definitions first.

Definition

Given any subset $A \subset \mathbb{R}$, A is **bounded above** *if there exists $u \in \mathbb{R}$ for which $a \le u$ for all $a \in A$. If $U \in \mathbb{R}$ is an upper bound for which $U \le u$ for* **every** *upper bound u of A, then we say that U is the* **least upper bound**. *There are two common notations for this:* $\operatorname{lub} A$ *and* $\sup A$.

Exercise: Prove that there is exactly one least upper bound, i.e. it's unique.

Similarly, we say A is **bounded below** if there exists $\ell \in \mathbb{R}$ for which $\ell \le a$ for all $a \in A$. If $L \in R$ is the greatest lower bound for A, then we write $L = \inf A$ or $L = \operatorname{glb} A$.

Exercise: Let

$$A = \{0, 1/2, 2/3, 3/4, 4/5, \ldots\}.$$

Show that $\inf A = 0$ and $\sup A = 1$.

In the language we shall explore later in this course, $\mathbb{R}$ is $\mathbb{Q}$ extended by including all convergent sequences of rationals. We can avoid sequences for the moment in a very neat way:

The Completeness Axiom: Every non-empty subset of $\mathbb{R}$ that is bounded above has exactly one **least upper bound** in $\mathbb{R}$.

Example

$A = \{x \in \mathbb{R} : x^2 < 2\}$, then $a = \sup A \in \mathbb{R}$ satisfies $a^2 = 2$.

Suppose $a^2 < 2$. Now

$$\left(a + \frac{1}{n}\right)^2 = a^2 + \frac{2a}{n} + \frac{1}{n^2} \leq a^2 + \frac{2a+1}{n}.$$

We know that there exists $n_0 \in \mathbb{N}$ such that

$$n_0 > \frac{2a+1}{2-a^2},$$

for otherwise $\mathbb{N}$ would be bounded. Hence

$$\left(a + \frac{1}{n_0}\right)^2 \leq a^2 + \frac{2a+1}{n_0} < a^2 + 2 - a^2 = 2.$$

But then $a + \frac{1}{n_0} < a$ which is nonsense.

Exercise Show that $a^2 > 2$ also leads to a contradiction.

Crucial Fact: The Completeness Axiom is **FALSE** in $\mathbb{Q}$.

Thus there is no rational number r which satisfies

$$r = \sup\{q \in \mathbb{Q} : q^2 < 2\}.$$

Theorem (Approximation Property)

Let S be a nonempty subset of $\mathbb{R}$ and let $U = \sup S$. Then, for every $a < U$, there exists $x \in S$ for which $a < x \le U$.

Proof.

If we had $x \le a$ for every $x \in S$, then a would be a smaller upper bound than $U = \sup S$, contradicting the definition of $\sup S$. Therefore $x > a$ for at least one $x \in S$. □

Theorem

$\mathbb{N}$ is unbounded above.

Proof.

If $\mathbb{N}$ were bounded above, then $U = \sup \mathbb{N} \in \mathbb{R}$, by the Completeness Axiom. By the Approximation Property, there would exist some $n \in \mathbb{N}$ for which $U - 1 < n$. But then $n + 1 > U$, i.e. U is **not** an upper bound, which is a contradiction. □

Theorem

Let $x \in \mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $n > x$.

Proof.

If this were not true, then $\mathbb{N}$ would be bounded above. □

Theorem (The Archimedean Property (or Axiom) of $\mathbb{R}$)

If $x > 0$ and $y \in \mathbb{R}$, then there is a positive integer n for which $nx > y$.

Proof.

There is a positive integer n exceeding y/x. □

It's time to return to some actual numbers. A real number of the form

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n},$$

where a_0 is a non-negative integer and $a_1, \ldots, a_n$ are integers satisfying $0 \le a_k \le 9$ is usually written as

$$r = a_0.a_1 a_2 \cdots a_n.$$

This is called a **finite decimal representation** of r.

Theorem (Arbitrarily accurate decimal approximations exist.)

Let $x \in \mathbb{R}_+$. Then for every integer $n \geq 1$ there exists a finite decimal $r_n = a_0.a_1 a_2 \cdots a_n$ such that

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

This is really our first algorithm, i.e. essentially a computer program: we call this a **constructive proof**.

Proof.

Let

$$S = \{n \in \mathbb{N} : 0 \le n \le x\}.$$

Then $a_0 = \sup S$ is a non-negative integer and we write $a_0 = [x]$, the greatest integer $\le x$. Thus

$$a_0 \le x < a_0 + 1.$$

Now let $a_1 = [10x - 10a_0]$, i.e. the greatest integer $\le 10x - 10a_0$. We have $0 \le 10x - 10a_0 = 10(x - a_0) < 10$, so $0 \le a_1 \le 9$ and

$$a_1 \le 10x - 10a_0 < a_1 + 1, \quad \text{or} \quad a_0 + \frac{a_1}{10} \le x < a_0 + \frac{a_1 + 1}{10}.$$

The algorithm then continues with $a_2 = [10^2 x - 10^2 a_0 - 10a_1]$. $\square$

The previous result tells us two important facts:

1. We can find infinitely many rational numbers between any two real numbers.
2. We can approximate any real number as closely as we wish.

We say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Example

Between any two real numbers there is an irrational number.

To see this, let the interval be (a, b). The trick is to consider the interval $(a + \sqrt{2}, b + \sqrt{2})$. We know that this interval contains a rational number, q say, i.e.

$$a + \sqrt{2} < q < b + \sqrt{2}.$$

Hence

$$a < q - \sqrt{2} < b.$$

Exercise: Prove that $q - \sqrt{2}$ is irrational if $q \in \mathbb{Q}$.

Exercise: Prove that there are infinitely many irrational numbers in (a, b).

One of the motivations for analysis was Cantor's new way of comparing infinite sets.

Definition

We say that any two sets X and Y have the same **cardinality** *if there is a bijection $f : X \to Y$.*

Thus two sets have the same cardinality if their elements can be paired up. In particularly, we say that a set X has cardinality $n \in \mathbb{N}$ if there is a bijection $f : X \to \{1, 2, \ldots, n\}$.

Cantor's insight was that we could use this definition to compare infinite sets too. For example, if we let $2\mathbb{Z}$ denote the even integers, i.e.

$$2\mathbb{Z} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\},$$

then $f : \mathbb{Z} \to 2\mathbb{Z}$ defined by $f(n) = 2n$, $n \in \mathbb{Z}$, is a bijection. Thus the even integers have the same cardinality as the integers, even though $2\mathbb{Z}$ is a proper subset of $\mathbb{Z}$.

Definition

We say that a set X is **countable** *if it's a finite set or if it has the same cardinality as the integers.*

Exercise: Show that the set of all odd integers is countable.

Theorem

The set

$$\mathbb{N} \times \mathbb{N} = \{(j, k) : j, k \in \mathbb{N}\}$$

is countable.

Proof.

Define $f(j, k) = 2^j 3^k$. Then f is injective (Why?) from $\mathbb{N} \times \mathbb{N}$ into a subset of $\mathbb{N}$. □

Theorem

$\mathbb{Q}$ *is countable.*

Proof.

The function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ given by $g(m, n) = m/n$ is sufficient, but I will give further details in the lecture. □

Theorem

$(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ *is not countable.*

Cantor's diagonal argument sketch.

If $(0,1)$ were countable, then it would be actually be a sequence $a_1, a_2, \ldots$, where each a_n is an infinite decimal

$$a_n = 0.a_{n,1}a_{n,2}\ldots, \qquad \text{for } n \in \mathbb{N}.$$

Now define a new $A = 0.A_1A_2\ldots \in \mathbb{R}$ by

$$A_n = \begin{cases} 1 & \text{if } a_{n,n} \neq 1, \\ 2 & \text{if } a_{n,n} = 1. \end{cases}.$$

Then A differs from a_n in the n^{th} decimal place, a contradiction. □

Nonexaminable fun: almost all reals are irrational.

To see this, we shall sketch some ideas from more advanced analysis. Let $q_1, q_2, \ldots$ denote the rational numbers in $(0,1)$. Given any $\epsilon > 0$, let

$$I_n = (q_n - \frac{\epsilon}{2^{n+1}}, q_n + \frac{\epsilon}{2^{n+1}}), \qquad \text{for } n \in \mathbb{N}.$$

Then $q_n \in I_n$ and the length of I_n is $L_n = \frac{\epsilon}{2^n}$. Thus $\mathbb{Q} \cap (0,1)$ is contained in $I_1 \cup I_2 \cup \cdots$ and the "length" of $\mathbb{Q} \cap (0,1)$ should be less than

$$L_1 + L_2 + L_3 + \cdots = \epsilon \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) = \epsilon.$$

Since $\epsilon > 0$ can be as small as we wish, we say that $\mathbb{Q} \cap (0,1)$ has **measure zero**, while $(0,1)$ has measure 1.