Real Analysis 2.5: From Sequences to Series

Brad Baxter
Birkbeck College, University of London

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You can download these slides and the lecture videos from my office server

`http://econ109.econ.bbk.ac.uk/brad/analysis/`

Recommended book: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.

Cauchy sequences are important for series too, so I shall begin with an example.

Example (Using the Cauchy condition)

Given any $x_0, x_1 \in \mathbb{R}$, we define the sequence

$$x_k = \frac{1}{2}\left(x_{k-1} + x_{k-2}\right), \qquad \text{for } k \geq 2.$$

Then

$$x_k - x_{k-1} = \frac{1}{2}\left(x_{k-1} + x_{k-2}\right) - x_{k-1} = -\frac{1}{2}\left(x_{k-1} - x_{k-2}\right),$$

or

$$\left|x_k - x_{k-1}\right| = \frac{1}{2}\left|x_{k-1} - x_{k-2}\right|.$$

Then

$$\begin{aligned}
|x_{N+p} - x_N| &\le |x_{N+p} - x_{N+p-1}| + \cdots + |x_{N+1} - x_N| \\
&\le \left(\frac{1}{2^p} + \frac{1}{2^{p-1}} + \cdots + \frac{1}{2}\right) |x_N - x_{N-1}| \\
&\le \left(\frac{1}{2^p} + \frac{1}{2^{p-1}} + \cdots + \frac{1}{2}\right) \frac{1}{2^{N-1}} |x_1 - x_0| \\
&\le \frac{1}{2^{N-1}} |x_1 - x_0| .
\end{aligned}$$

Now, given any $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$\frac{1}{2^{N-1}} |x_1 - x_0| < \epsilon$$

Hence (x_k) is a Cauchy sequence, and therefore convergent. [It can be shown that the solution is $x_n = A + B(-1/2)^n$, where $A = (x_0 + 2x_1)/3$ and $B = x_0 - A$.]

Example (Solving $x_{k+1} = (x_k + x_{k-1})/2$)

We saw earlier that

$$x_{k+1} - x_k = C\left(x_k - x_{k-1}\right),$$

where $C = -1/2$. Hence

$$\begin{aligned} x_k - x_{k-1} &= C\left(x_{k-1} - x_{k-2}\right) \\ &= C^2\left(x_{k-2} - x_{k-3}\right) \\ &= \cdots \\ &= C^{k-1}\left(x_1 - x_0\right). \end{aligned}$$

Exercise: Use

$$x_k - x_{k-1} = C^{k-1}(x_1 - x_0)$$

to show that (later you will see this is a **telescoping sum**)

$$\sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0$$

and (we will see geometric series later this lecture), recalling that $C = -1/2$,

$$\sum_{k=1}^{n} C^{k-1} = \frac{2}{3}\left(1 - (-1/2)^n\right).$$

Hence show that $x_n = A + B(-1/2)^n$, where $A = (x_0 + 2x_1)/3$ and $B = x_0 - A$.

Theorem (Sums of Powers)

You should know that

$$\sum_{k=0}^{n} k = \frac{1}{2} n(n+1)$$

and

$$\sum_{k=0}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

and I cannot resist stating the beautiful fact that

$$\sum_{k=0}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2 .$$

You have probably all seen $\sum_0^n k$ and $\sum_0^n k^2$, but where do they come from? The answer lies in a fascinating borderland between series and the origins of calculus.

Definition

The **forward difference operator** Δ is defined by

$$\Delta a_n = a_{n+1} - a_n.$$

Example

1. If $a_n = c$, for all n, then $\Delta a_n = 0$.
2. If $a_n = n$, then $\Delta a_n = n + 1 - n = 1$.
3. $\Delta n^2 = (n+1)^2 - n^2 = 2n + 1$
4. $\Delta n^3 = (n+1)^3 - n^3 = 3n^2 + 3n + 1$.

Example ($\sum_{k=0}^{n} k$)

If $S_n = 0 + 1 + 2 + \cdots + n$, then $\Delta S_n = S_{n+1} - S_n = n + 1$. Then

$$\Delta \left(S_n - An^2 - Bn \right) = 0,$$

if

$$n + 1 - A(2n + 1) - B = 0, \qquad \text{for all } n,$$

or $A = B = 1/2$. Now $\Delta(S_n - An^2 - Bn) = 0$ implies that $S_n - An^2 - Bn = c$, for some constant c, but setting $n = 0$ implies $c = 0$.

Theorem ($\sum_{k=0}^{n} k^2$)

Let $T_n = 0^2 + 1^2 + 2^2 + \cdots + n^2$. *Then*

$$T_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

Proof.

Now

$$\Delta T_n = T_{n+1} - T_n = (n+1)^2 = n^2 + 2n + 1.$$

We must therefore find constants P, Q and R for which

$$\Delta\left(Pn^3 + Qn^2 + Rn\right) = 2n^2 + 2n + 1,$$

i.e. equating coefficients of powers of n, we have

$$P(3n^2 + 3n + 1) + Q(2n + 1) + R = n^2 + 2n + 1.$$

Hence $3P = 1$, $3P + 2Q = 2$ and $P + Q + R = 1$. □

Example ($\sum_{k=0}^{n} k^3$)

Let $U_n = \sum_{k=0}^{n} k^3$. Then

$$\Delta U_n = (n+1)^3 = n^3 + 3n + 3n + 1$$

and we need constants $A_1, A_2, A_3, A_4 \in \mathbb{R}$ for which

$$\Delta \left(A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n \right) = n^3 + 3n^2 + 3n + 1.$$

Now

$$LHS = A_4(4n^3+6n^2+4n+1) + A_3(3n^2+3n+1) + A_2(2n+1) + A_1$$

so that, equating coefficients of powers of n,

$$\begin{aligned} 4A_4 &= 1, \\ 6A_4 + 3A_3 &= 3, \\ 4A_4 + 3A_3 + 2A_2 &= 3, \\ A_4 + A_3 + A_2 + A_1 &= 1. \end{aligned}$$

Exercise: Hence show that

$$U_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = S_n^2,$$

where $S_n = \sum_{k=0}^{n} k$.
Note: Sadly the beautiful fact that $U_n = S_n^2$ does not extend to higher sums of powers.