Real Analysis 2: Sequences

Brad Baxter
Birkbeck College, University of London

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You can download these slides and the lecture videos from my office server

http://econ109.econ.bbk.ac.uk/brad/analysis/

Recommended book: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.

Definition

We say that a sequence of real numbers $a_1, a_2, \ldots$ is an **increasing sequence** *if*

$$a_1 \le a_2 \le a_3 \le \cdots ,$$

i.e. $a_k \le a_{k+1}$ *for all* $k \in \mathbb{N}$.

Example

Here are 3 increasing sequences:

1. The constant sequence $a_k = 1$, for all $k \in \mathbb{N}$, is an increasing sequence, albeit a rather boring example (think of increasing as really meaning non-decreasing, if it helps).
2. $a_k = 1 - \frac{1}{k}$, for $k \in \mathbb{N}$.
3. $a_k = 10^k$, for $k = 0, 1, 2, \ldots$ (i.e. the index can start at 0).

Definition (ϵ, N_ϵ definition of convergence)

A real sequence (a_n) is **convergent** *with limit $a \in \mathbb{R}$ if, for any positive $\epsilon > 0$, there exists an integer $N \equiv N_\epsilon$ for which*

$$|a_n - a| < \epsilon$$

for all $n \geq N$. We write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$, as $n \to \infty$.

If a sequence isn't convergent, then we say it's divergent.

Example

$a_k = 10^k$ and $b_k = (-1)^k$ are divergent sequences.

Example

If $a_n = 1$ for all $n \in \mathbb{N}$, then $a_n \to 1$, as $n \to \infty$. The key point is that $a_n - 1 = 0$, for all n, so that, given any $\epsilon > 0$, we have $|a_n - 1| < \epsilon$ for all n.

Example

If $a_n = 1 - 1/n$, for $n \in \mathbb{N}$, then $a_n \to 1$, as $n \to \infty$. Indeed, given any $\epsilon > 0$, we have

$$|a_n - 1| = \frac{1}{n} < \epsilon$$

for all integer n such that $n > 1/\epsilon$, i.e. for all sufficiently large integer n, by the Archimedean Property of $\mathbb{R}$.

Theorem

A real increasing sequence (a_n) *that is bounded above is convergent and* $\lim_{n\to\infty} a_n = \sup a_n$*.*

Proof.

Choose any $\epsilon > 0$. If $a = \sup a_n$, then there must be at least one member of the sequence (a_n), a_N say, in the interval $(a-\epsilon, a)$, for otherwise a would not be the least upper bound. Since (a_n) is an increasing sequence, we have

$$a - \epsilon < a_N \le a_{N+1} \le a_{N+2} \le \cdots \le a.$$

Thus $\lim_{n\to\infty} a_n = a$. $\square$

Example

Let $b_n = 1 + n^{-1}$. We shall prove that $b_n \to 1$ as $n \to \infty$. Thus, given any $\epsilon > 0$,

$$|b_n - 1| = |n^{-1}| < \epsilon$$

when $n \geq N$ and $N > \epsilon^{-1}$.

Exercise: Are these sequences convergent or divergent? Find their limits if convergent.

1. $a_n = (-1)^n$, for $n \in \mathbb{N}$.
2. $a_n = \frac{(-1)^n}{n^2}$, for $n \in \mathbb{N}$.
3. $a_n = \frac{\sin(300n)}{n}$, for $n \in \mathbb{N}$.
4. $$a_n = \begin{cases} 1 & \text{if } n = 10^m \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Definition

The **absolute value** $|x|$ *of* $x \in \mathbb{R}$ *is defined by*

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Theorem

If $a \geq 0$, *then* $|x| \leq a$ *if and only if* $-a \leq x \leq a$.

Theorem (The Triangle Inequality)

If $a, b \in \mathbb{R}$, *then* $|a+b| \le |a| + |b|$.

Proof.

We have

$$-|a| \le a \le |a| \quad \text{and} \quad -|b| \le b \le |b|,$$

and adding these inequalities gives

$$-(|a| + |b|) \le a + b \le (|a| + |b|)$$

which implies

$$|a+b| \le |a| + |b|.$$

□

Theorem (Alternative form of Triangle Inequality)

If $a, b \in \mathbb{R}$, then

$$|a-b| \geq \Big||a|-|b|\Big|.$$

Proof.

If we let $x = a - b$ and $y = b$, then $x + y = a$ and the triangle inequality $|x+y| \leq |x| + |y|$ becomes $|a| \leq |a-b| + |b|$ or

$$|a| - |b| \leq |a-b|.$$

Further, if we let $x = b - a$ and $y = a$, then $x + y = b$ and the triangle inequality becomes $|b| \leq |b-a| + |a|$, or

$$|b| - |a| = -\left(|a| - |b|\right) \leq |a-b|.$$

$\square$

Theorem

A convergent sequence is bounded.

Proof.

If $a_1, a_2, \ldots$ is a convergent sequence with limit a, then there exists $N \in \mathbb{N}$ for which

$$|a_n - a| < 1,$$

for all $n \geq N$. In particular,

$$a - 1 \leq a_n \leq a + 1,$$

for all $n \geq N$. Thus the sequence is bounded. □

Theorem (Sequence Addition)

Suppose $a_n \to a$ and $b_n \to b$, as $n \to \infty$. Then $a_n + b_n \to a + b$.

Proof.

Given any $\epsilon > 0$, choose a positive integer N so large that both

$$|a_n - a| < \epsilon/2 \qquad \text{and} \qquad |b_n - b| < \epsilon/2.$$

Then

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) - (b_n - b)| \\ &\le |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$\square$

Theorem (Sequence Multiplication)

Suppose $a_n \to a$ *and* $b_n \to b$*, as* $n \to \infty$*. Then* $a_n b_n \to ab$*.*

Proof.

Let $u_n = a_n - a$ and $v_n = b_n - b$. Then $u_n \to 0$ and $v_n \to 0$, and $a_n = a + u_n$, $b_n = b + v_n$. Further

$$a_n b_n - ab = (a + u_n)(b + v_n) - ab = a v_n + b u_n + u_n v_n.$$

Now, given any $\alpha \in (0,1)$, there exists $N \in \mathbb{N}$ for which $|u_n| < \alpha$ and $|v_n| < \alpha$ when $n \geq N_\alpha$. Further, $|u_n v_n| < \alpha^2 < \alpha$, for all $n \geq N_\alpha$, since $\alpha \in (0,1)$. Hence

$$\begin{aligned} |a_n b_n - ab| &= |(a + u_n)(b + v_n) - ab| \\ &= |a v_n + b u_n + u_n v_n| \\ &\leq |a v_n| + |b u_n| + |u_n v_n| < (|a| + |b| + 1)\alpha. \end{aligned}$$

Hence, given any $\epsilon > 0$, choose $\alpha < \frac{\epsilon}{|a|+|b|+1}$ and $n \geq N_\alpha$. □

Suppose $a_n \to a$, $b_n \to b$ and $b \neq 0$. We shall soon prove that $a_n/b_n \to a/b$, but we shall need a simple example first.

Example

Let $b_n \to b$ and initially suppose that $b > 0$. Then $b_n > 0$ for all sufficiently large n, i.e. a real sequence with positive limit is ultimately positive.

The key point is the definition of convergence: given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|b_n - b| < \epsilon$. In particular, if we choose $\epsilon = b/2$, then

$$|b_n - b| < b/2,$$

i.e.

$$-b/2 < b_n - b < b/2,$$

which implies

$$b/2 < b_n < 3b/2,$$

when $n \geq N$.

When the limit b is negative, we have $b = -|b|$ and setting $\epsilon = |b|/2$ implies

$$-|b|/2 < b_n - b < |b|/2,$$

for (say) $n \geq N$. But we can rewrite this as

$$-|b|/2 < b_n + |b| < |b|/2,$$

whence

$$-3|b|/2 < b_n < -|b|/2,$$

and $b_n < -|b|/2$ implies that $|b_n| > |b|/2$.

Theorem (Sequence Division)

Let $a_n \to a$, $b_n \to b$ and suppose that $b \neq 0$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Proof.

There exists $M \in \mathbb{N}$ such that $|b_n| > |b|/2$ for $n \geq M$. Hence

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|bb_n|},$$

and, for $n \geq M$, $|b_n| > |b|/2$ implies $\frac{1}{|bb_n|} < \frac{2}{|b|^2}$, so that

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|bb_n|} < \left(\frac{2}{|b|^2} \right) |b - b_n|$$

and this can be made arbitrarily small for all sufficiently large n. Thus the sequence $c_n = b_n^{-1}$ is well defined for $n \geq M$ and $c_n \to b^{-1}$. Finally, $a_n c_n \to ab^{-1}$. $\square$

Example

Let's use our new knowledge of sequence arithmetic. Recall that $w_n = (3n+1)/(n+4)$, for $n \ge 1$. Then, dividing numerator and denominator by n, we obtain

$$w_n = \frac{3 + n^{-1}}{1 + 4n^{-1}} \equiv \frac{a_n}{b_n}.$$

Now $a_n \to 3$ and $b_n \to 1 \neq 0$ (by the Archimedean Property of $\mathbb{R}$), so $w_n \to 3$.

Theorem (Bolzano–Weierstrass)

Any sequence $(x_n)_{n\in\mathbb{N}}$ *contained in the interval* $[0,1]$ *has a convergent subsequence* $x_{n_1}, x_{n_2}, \ldots$.

Proof.

Define $a_0 = 0$ and $b_0 = 1$. At least one of the intervals $[0, 1/2]$ and $[1/2, 1]$ must contain infinitely many members of the sequence, $[a_1, b_1]$ say. Note that $b_1 - a_1 = 1/2$.

Similarly, at least one of the intervals $[a_1, a_1 + 1/4]$, $[a_1 + 1/4, b_1]$ must contain infinitely many members of the sequence, $[a_2, b_2]$ say, where $b_2 - a_2 = 2^{-2}$.

Repeating this construction, we obtain an increasing sequence (a_k) and a decreasing sequence (b_k) in $[0,1]$ for which $b_k - a_k = 2^{-k}$ and $[a_k, b_k]$ contains infinitely many members of (x_n); pick any member x_{n_k} of the sequence in $[a_k, b_k]$.

$\square$

Theorem (Bolzano–Weierstrass)

Any bounded sequence has a convergent subsequence.

Proof.

If the sequence (x_n) is bounded, then it's contained in a bounded interval, $[a, b]$ say. If we define

$$y = \frac{x-a}{b-a}, \qquad x \in \mathbb{R},$$

then this linear function maps the interval $[a, b]$ onto $[0, 1]$ and it's a bijection with inverse

$$x = a + (b-a)\, y.$$

Now apply the previous version of Bolzano–Weierstrass to $y_n = (x_n - a)/(b-a)$ to obtain a convergent subsequence (y_{n_k}), and hence (x_{n_k}). □

Example

Let $a > 1$. We shall prove that $a^{1/n} \to 1$, as $n \to \infty$. We know that $a^{1/n} > 1$, for all n (Why?). Hence we can write

$$a^{1/n} = 1 + d_n, \qquad n \geq 1,$$

and $d_n > 0$. The binomial theorem implies that

$$\begin{aligned} a &= (1 + d_n)^n \\ &= 1 + nd_n + \frac{n(n-1)}{2!} d_n^2 + \cdots + d_n^n \\ &> nd_n, \end{aligned}$$

since every term in the binomial expansion is positive. Thus

$$d_n < \frac{a}{n}, \qquad n \geq 1,$$

and therefore $d_n \to 0$, as $n \to \infty$, which implies $a^{1/n} \to 1$.

Example

We shall now show that $n^{1/n} \to 1$, as $n \to \infty$, using a slight modification of the proof technique of the previous example. We again write $n^{1/n} = 1 + d_n$, and observe that $d_n > 0$. Hence the binomial theorem implies that

$$\begin{aligned} n &= (1 + d_n)^n \\ &= 1 + nd_n + \frac{n(n-1)}{2!} d_n^2 + \cdots + d_n^n \\ &> \frac{n(n-1)}{2!} d_n^2, \end{aligned}$$

since every term in the expansion is positive, whence

$$d_n^2 < \frac{2}{n-1},$$

for $n \geq 2$. In particular, we have shown that $d_n \to 0$, which implies that $n^{1/n} \to 1$, as $n \to \infty$.

We will study the exponential function later in this course, but we shall borrow one result from a future lecture here: if $x > 0$ and

$$S_m(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^m}{m!},$$

then $S_m(x)$ is a bounded increasing sequence and its limit is the exponential function $\exp(x)$.

Theorem (Exponential decay beats linear growth)

We have $\lim_{n\to\infty} \frac{n}{\exp(cn)} = 0$, *for any* $c > 0$.

Proof.

If $x > 0$, then

$$\exp(x) > S_2(x) > \frac{x^2}{2}$$

so that

$$\frac{n}{\exp(cn)} < \frac{n}{c^2 n^2/2} = \frac{2}{c^2 n},$$

and the upper bound tends to zero as $n \to \infty$. $\square$

Exercise: Show that, for $c > 0$, we have

$$\frac{n}{\exp(cn)} < \frac{n}{c^3 n^3/6} = \frac{6c^{-3}}{n^2}.$$

Exercise: Show that

$$\frac{n^2}{\exp(cn)} < \frac{6c^{-3}}{n}.$$

Exercise: Show that

$$\frac{n^{100}}{\exp(n)} < \frac{101!}{n}.$$

Hence $n^{100}/\exp(n) \to 0$ as $n \to \infty$. However,

$$\frac{10^{100}}{\exp(10)} \approx 4.54 \times 10^{95}.$$

Is this a contradiction?

Theorem

Let $A > 0$ and choose $x_1 > \sqrt{A}$. Then the sequence defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right), \quad n \in \mathbb{N},$$

is decreasing and $\lim x_n = \sqrt{A}$.

Let's try it: if $A = 3$ and $x_1 = 2$, then we obtain:

$$\begin{aligned} x_2 &= 1.750000000000000 \\ x_3 &= 1.732142857142857 \\ x_4 &= 1.732050810014727 \\ x_5 &= 1.732050807568877, \end{aligned}$$

and x_5 is correct to the number of digits displayed.

We can make life easier by defining $x_n = \sqrt{A} y_n$.

Exercise: $y_{n+1} = \frac{1}{2}\left(y_n + \frac{1}{y_n}\right)$.

Theorem

Choose $y_1 > 1$. Then the sequence defined by

$$y_{n+1} = \frac{1}{2}\left(y_n + \frac{1}{y_n}\right), \quad n \in \mathbb{N},$$

is decreasing and $\lim y_n = 1$. Further, if $e_n = y_n - 1 < 1$, then

$$e_{n+1} = \frac{\frac{1}{2} e_n^2}{1 + e_n},$$

i.e. the error ultimately decreases quadratically: the number of correct digits ultimately doubles on each step.

Exercise: If e_1 is large, then $e_2 \approx e_1/2$.

We're going to need a simple inequality.

Theorem

Let $a, b \in \mathbb{R}$. Then

$$\frac{a^2 + b^2}{2} \geq ab,$$

with equality if and only if $a = b$.

Proof.

We have

$$0 \leq (a - b)^2 = a^2 + b^2 - 2ab,$$

with equality if and only if $a = b$. $\square$

Theorem (AM-GM Inequality)

If $x \geq 0$ and $y \geq 0$, then

$$\frac{x+y}{2} \geq \sqrt{xy},$$

with equality if and only if $x = y$.

Proof.

Let $x = a^2$ and $y = b^2$ in the previous theorem. $\square$

Now we can prove that the sequence $y_1 > 1$ and $y_{n+1} = (y_n + y_n^{-1})/2$, for $n \in \mathbb{N}$, is decreasing with limit 1.

Proof.

Now $y_1 > 1$ implies $1/y_1 < 1$, so that

$$y_2 = \frac{1}{2}\left(y_1 + \frac{1}{y_1}\right) < \frac{1}{2}(y_1 + 1) < \frac{1}{2}(y_1 + y_1) = y_1.$$

Further, by the AM-GM inequality,

$$y_2 = \frac{1}{2}\left(y_1 + \frac{1}{y_1}\right) > \sqrt{y_1 \cdot \frac{1}{y_1}} = 1.$$

We now repeat the argument to show that $1 < y_3 < y_2$, etc. $\square$

We need to borrow a result from next week's lecture:

Theorem (Sum of Geometric Series)

If $|a| < 1$, then the sequence defined by

$$S_n = 1 - a + a^2 - a^3 + \cdots + (-1)^n a^n$$

converges to $1/(1+a)$.

Theorem

Let $e_n = y_n - 1$. Then (e_n) is a positive decreasing sequence with limit zero. Further, if $e_n < 1$, then

$$e_{n+1} = \frac{\frac{1}{2}e_n^2}{1 + e_n}.$$

Proof.

Substituting $y_n = 1 + e_n$ in the iteration $y_{n+1} = (y_n + y_n^{-1})/2$, we use the geometric series when $e_n < 1$ to obtain

$$
\begin{aligned}
1 + e_{n+1} &= \frac{1}{2}\left(1 + e_n + \frac{1}{1 + e_n}\right) \\
&= \frac{1}{2}\left(1 + e_n + 1 - e_n + e_n^2 - e_n^3 + e_n^4 - \cdots\right) \\
&= 1 + \frac{1}{2}\left(e_n^2 - e_n^3 + e_n^4 - \cdots\right) \\
&= 1 + \frac{1}{2}e_n^2\left(1 - e_n + e_n^2 - \cdots\right) \\
&= 1 + \frac{\frac{1}{2}e_n^2}{1 + e_n}.
\end{aligned}
$$

$\square$

There is a clever way to characterize convergent sequences.

Definition

A real sequence (x_n) *is a* **Cauchy sequence** *if, given any* $\epsilon > 0$, *there exists* $N \equiv N_\epsilon \in \mathbb{N}$ *for which*

$$|x_m - x_n| < \epsilon$$

when $m, n \geq N$.

Theorem

A Cauchy sequence is bounded.

Proof.

There exists N such that, for $m \geq N$, we have

$$|x_m - x_N| < 1,$$

i.e.

$$x_N - 1 \leq x_m \leq x_N + 1, \qquad \text{for all } m \geq N.$$

Theorem

A convergent sequence is a Cauchy sequence.

Proof.

If $x_n \to a$, then, given any $\epsilon > 0$, there exists $N \equiv N_\epsilon \in \mathbb{N}$ for which

$$|x_n - a| < \frac{\epsilon}{2}$$

for $n \ge N$. Hence, by the Triangle inequality,

$$|x_m - x_n| \le |x_m - a| + |a - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for $m, n \ge N$. □

Theorem

A Cauchy sequence is a convergent sequence.

Proof.

We already know that a Cauchy sequence (x_n) is bounded so, by the Bolzano–Weierstrass theorem, there exists a convergent subsequence (x_{n_k}), with limit a say. We can therefore choose $N \in \mathbb{N}$ such that

$$|x_{n_k} - a| < \epsilon/2,$$

for $n_k \geq N$ **and** (because (x_n) is a Cauchy sequence)

$$|x_{n_k} - x_n| < \epsilon/2,$$

for $n \geq N$. Hence

$$|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

The construction of $\mathbb{R}$ via Cauchy sequences in $\mathbb{Q}$

If $Q = (q_n)$ and $R = (r_n)$ are any two Cauchy sequences in $\mathbb{Q}$ for which $q_n - r_n \to 0$, as $n \to \infty$, then we write $Q \sim R$. It's not difficult to show that $\sim$ defines an equivalence relation on the set of all Cauchy sequences in $\mathbb{Q}$. The real numbers are exactly the equivalence classes. If $\langle Q \rangle$ denotes the equivalence class of rational Cauchy sequences sharing the limit of sequence Q, then we define addition and multiplication on the equivalence classes via

$$\langle Q \rangle + \langle R \rangle = \langle Q + R \rangle \quad \text{and} \quad \langle Q \rangle \cdot \langle R \rangle = \langle Q \cdot R \rangle.$$