Real Analysis 3.5: Power Series

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You can download these slides and the lecture videos from my office server

http://econ109.econ.bbk.ac.uk/brad/analysis/

Recommended books:

Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.

J. C. Burkill (1978), "A First Course in Mathematical Analysis", Cambridge University Press.

A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where (a_n) is a complex sequence and $z \in \mathbb{C}$. This is still an infinite series, so it's convergent if and only if its partial sums

$$s_N = \sum_{n=0}^{N} a_n z^n, \qquad \text{for } N = 0, 1, \ldots,$$

form a convergence sequence.

Theorem

Suppose $w \in \mathbb{C}$, $w \neq 0$ and $\sum_{n=0}^{\infty} a_n w^n$ is convergent. Then $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for all $|z| < |w|$.

Proof.

If $\sum_{n=0}^{\infty} a_n w^n$ is convergent, then $a_n w^n \to 0$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $|a_n w^n| \leq 1$ for $n \geq N$. Therefore, if $|z| < |w|$ and $n \geq N$, we obtain

$$|a_n z^n| = |a_n w^n| \, |(z/w)^n| \leq \left|\frac{z}{w}\right|^n.$$

Thus the series is absolutely convergent for all $|z| < |w|$:

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n z^n| &= \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} |a_n z^n| \\ &\leq \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} \left|\frac{z}{w}\right|^n\end{aligned}$$

Now let's study the error:

$$E_M(z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{M-1} a_n z^n = \sum_{n=M}^{\infty} a_n z^n.$$

If $|z/w| \le A < 1$ and $M \ge N$, then

$$|E_M(z)| \le \sum_{n=M}^{\infty} |a_n z^n| \le \sum_{n=M}^{\infty} \left|\frac{z}{w}\right|^n \le \sum_{n=M}^{\infty} A^n = \frac{A^n}{1-A}.$$

In other words the error can be made **uniformly small** in $|z| \le A|w|$, where $A \in (0,1)$.

Theorem (Disc and Radius of Convergence)

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and let S be the set of z for which this converges and let

$$R = \sup\{|z - z_0| : z \in S\}.$$

We write $R = +\infty$ if S is unbounded. The power series is absolutely convergent if $|z - z_0| < R$ and divergent for $|z - z_0| > R$. The set

$$\{z \in \mathbb{C} : |z - z_0| < R\}$$

is called the **disc of convergence** *and R is called the* **radius of convergence**. *We call $\{z \in \mathbb{C} : |z| = R\}$ the* **circle of convergence**.

Proof.

We know S is nonempty because $0 \in S$. If $R = \infty$, then the power series is convergent for all $z \in \mathbb{C}$, and hence absolutely convergent for all $z \in \mathbb{C}$.

If $R > 0$, then the power series converges absolutely for $|z| < R$, by the previous theorem.

Further, if the power series converged for $|w| > R$, then it would converge for any z satisfying $R < |z| < |w|$, which contradicts the definition of R. □

Nothing has been proved about convergence or divergence on the circle of convergence.

Theorem (Ratio test for convergence of power series)

If the ratios $|a_{n+1}/a_n| \to L$ as $n \to \infty$, then the radius of convergence $R = 1/L$.

Proof.

By the Ratio test we have absolute convergence if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}z^{n+1}}{a_n z^n} \right| < 1,$$

i.e. if $|z| < 1/L$. Further, if $|z| > 1/L$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}z^{n+1}}{a_n z^n} \right| > 1,$$

so the nth term $a_n z^n$ does not tend to zero. □

Example (Warning: the Ratio test might not be applicable!)

Suppose

$$f(z) = 1 + z^2 + z^4 + z^6 + \cdots ,$$

i.e. $a_n = 1$ for even n, but $a_n = 0$ for odd n. Then the ratio $|a_{n+1}/a_n|$ is not even defined for odd n, so we cannot use the Ratio test here. We can, however, use the Root test, since

$$|a_n z^n|^{1/n} = |a_n|^{1/n} |z| \le |z|,$$

so the series is convergent if $|z| < 1$.

Theorem (Root test criterion for power series)

If $L > 0$ satisfies $|a_n|^{1/n} \leq L$ for all sufficiently large n, then the radius of convergence satisfies $R \geq 1/L$.

Proof.

We have

$$|a_n z^n|^{1/n} = |a_n|^{1/n}|z| \leq L|z|,$$

so the power series is absolutely convergent for $|z| < 1/L$. □

Optional extra

Theorem (Radius of convergence formula)

Let

$$L_n = \sup\{|a_k|^{1/k} : k \geq n\}.$$

Then (L_n) *is a decreasing sequence of non-negative numbers, so it's convergent: let* $L = \lim_{n\to\infty} L_n$*. Then the radius of convergence is given by* $R = 1/L$ *if* $L > 0$*, while* $R = \infty$ *if* $L = 0$*.*

Proof.

Given any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ for which $L \leq L_n < L + \epsilon$ for all $n \geq N_\epsilon$. Hence, if $n \geq N_\epsilon$, then

$$|a_n z^n|^{1/n} = |a_n|^{1/n}|z| < (L + \epsilon)|z|.$$

Thus we require $|z| < 1/(L + \epsilon)$ for convergence. Since $\epsilon > 0$ was arbitrary, we require $|z| < 1/L$. Conversely, if $L|z| > 1$, then $L_n|z| > 1$, for all n, and we have divergence. □

Example (Root test is stronger than Ratio test)

The power series

$$f(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

is absolutely convergent for $|z| < 1$ and has radius of convergence $R = 1$. Hence $g(z) = f(z) + f(z^3) = 1/(1-z) + 2/(1-z^3)$, for $|z| < 1$, and

$$g(z) = \sum_{k=0}^{\infty} a_k z^k$$

where $a_k = 2$ when k is an integer multiple of 3, but is otherwise equal to 1. Thus $a_k/a_{k-1} = 2$ when k is a multiple of 3, so the Ratio test cannot prove that this power series is convergent. The root test does, since $2^{1/n} \to 1$, as $n \to \infty$.