Real Analysis 3: Series

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You can download these slides and the lecture videos from my office server

http://econ109.econ.bbk.ac.uk/brad/analysis/

Recommended book: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.

1700s: Approximation of trigonometric and other functions by polynomials, e.g.

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4.$$

The higher the polynomial degree, the better the approximation (usually). This led to infinite series, e.g.

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r},$$

which is valid for $|r| < 1$ but **not** for $|r| > 1$, as we shall soon see. The case $r = -1$ is particularly worrying: the RHS makes sense and is equal to $1/2$, but we can write the LHS as

$$1 - 1 + (1 - 1) + (1 - 1) + \cdots$$

which looks like it should give zero, while different brackets yield

$$1 + (-1 + 1) + (-1 + 1) \cdots$$

which looks like it should be equal to 1! It's not a good sign.

Definition

Let $a_1, a_2, \ldots$ be any real sequence. We define the n^{th} **partial sum**

$$S_n = a_1 + a_2 + \cdots + a_n, \qquad \text{for } n \in \mathbb{N}.$$

If the sequence (S_n) is convergent, with limit S, then we say that the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent with limit S. If (S_n) isn't convergent, then we say that the series is divergent.

NB I've labelled the sequence $a_1, a_2, \ldots$, but the same definition works if our sequence is labelled $a_0, a_1, \ldots$.

Example

Let $a_n = (-1)^n$, for non-negative integer n. Then

$$S_n = 1 - 1 + 1 - 1 + \cdots + (-1)^n, \qquad n \ge 0.$$

Thus $S_n = 0$ when n is odd, but $S_n = 1$ when n is even. We see that the sequence (S_n) is divergent, so the infinite series

$$\sum_{n=0}^{\infty} (-1)^n$$

is divergent too.

Theorem (Cauchy convergence condition for series)

If (a_k) is a real sequence and

$$S_n = \sum_{k=1}^{n} a_k, \qquad n \in \mathbb{N},$$

are its partial sums, then the series $\sum a_k$ is convergent if and only if (S_n) is a Cauchy sequence. In other words, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ for which

$$\left| \sum_{k=n}^{n+p} a_k \right| < \epsilon$$

for any $n \geq N$ and $p \in \mathbb{N}$.

Proof.

The partial sums are convergent if and only if they form a Cauchy sequence. □

Theorem

If $\sum a_k$ and $\sum b_k$ are convergent series and $P, Q \in \mathbb{R}$, then $\sum (Pa_k + Qb_k)$ is also convergent and

$$\sum_{k=1}^{\infty} (Pa_k + Qb_k) = P \sum_{k=1}^{\infty} a_k + Q \sum_{k=1}^{\infty} b_k.$$

Proof.

The partial sums

$$A_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad B_n = \sum_{k=1}^{n} b_k$$

are both convergent sequences, with limits $A = \lim_{n\to\infty} A_n = \sum_{k=1}^{\infty} a_k$ and $B = \lim_{n\to\infty} B_n = \sum_{k=1}^{\infty} b_k$. Hence $(PA_n + QB_n)$ is a convergent sequence with limit $PA + QB$. □

Theorem (Telescoping Series)

Let $a_k = b_{k+1} - b_k$, for $k \in \mathbb{N}$. Then $\sum a_k$ is convergent if and only if the sequence (b_k) is convergent, in which case

$$\sum_{k=1}^{\infty} a_k = \lim_{k\to\infty} b_k - b_1.$$

Proof.

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_{k+1} - b_k) = b_{n+1} - b_1.$$

□

Example (Telescoping series)

Let

$$a_k = \frac{1}{k(k+1)}, \qquad k \in \mathbb{N}.$$

This is a telescoping series in disguise, since (Exercise)

$$a_k = \frac{1}{k} - \frac{1}{k+1} \equiv b_k - b_{k+1},$$

for $k \in \mathbb{N}$. Since $b_k = 1/k \to 0$, as $k \to \infty$, the series $\sum_{k=1}^{n} a_k \to b_1 = 1$, as $n \to \infty$.

Example (Comparison of series)

We can use the convergence of $\sum a_k$ when

$$a_k = \frac{1}{k(k+1)}, \qquad k \in \mathbb{N},$$

to prove the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

The key point is **comparison**:

$$\frac{k(k+1)}{k^2} = 1 + (1/k) \to 1, \qquad \text{as } k \to \infty,$$

so we can choose $K \in \mathbb{N}$ for which $1/k^2 \le 2a_k$, for $k \ge K$.

Theorem (Comparison Test 1)

Suppose $0 \leq a_k \leq b_k$ *for all* $k \in \mathbb{N}$*. If the series* $\sum_{k=1}^{\infty} b_k$ *is convergent, then so is the series* $\sum_{k=1}^{\infty} a_k$*.*

Proof.

The partial sums

$$B_n = \sum_{k=1}^{n} b_k$$

are increasing with limit B, say, since the series $\sum b_k$ is convergent. Then the partial sums

$$A_n = \sum_{k=1}^{n} a_k$$

satisfy $A_n \leq B_n \leq B$ and (A_n) is a bounded increasing sequence, since the a_k are non-negative. Hence the partial sums (A_n) are convergent, with limit A satisfying $A \leq B$. □

There are several variants of the Comparison Test.

Theorem (Comparison Test 2)

Let (a_k) and (b_k) be sequences of positive numbers for which there exists $N \in \mathbb{N}$ and a constant $C > 0$ such that

$$0 \le a_k \le Cb_k, \qquad \text{for } k \ge N.$$

If $\sum b_k$ is convergent, then $\sum a_k$ is convergent.

Proof.

This is an easy exercise. □

Definition

A series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ is convergent. A convergent series that is not absolutely convergent is called **conditionally convergent**

Absolutely convergent series are easier to deal with. Conditionally convergent series rely on delicate cancellation to achieve convergence. The series $\sum_{k=1}^{\infty} |a_k|$ is convergent if and only if

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

Example (Conditionally convergent series)

We shall see later that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$$

but

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

This this is a conditionally convergent series, not an absolutely convergent series.

Theorem (Absolute convergence implies convergence)

If $\sum |a_k| < \infty$, then $\sum a_k$ is convergent.

Proof.

We use the Cauchy condition for series and the triangle inequality:

$$\left| \sum_{k=n}^{n+p} a_k \right| \le \sum_{k=n}^{n+p} |a_k|.$$

□

Example (Absolute convergent series)

We have already seen that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

Hence

$$\sum_{k=1}^{\infty} \frac{sin(2^k \pi/3)}{k^2} < \infty$$

is also convergent.

Theorem (The Non-null test)

If $\sum a_k$ is a convergent series, then $a_k \to 0$ as $k \to \infty$.

Proof.

If the partial sums

$$S_n = \sum_{k=1}^{n} a_k, \qquad \text{for } n \in \mathbb{N},$$

form a convergent sequence, then we must have $S_n \to S$, say. But then $a_n = S_n - S_{n-1} \to S - S = 0$. □

Example

If

$$a_k = \frac{k^2 - 1}{k^2 + 1}, \qquad k \in \mathbb{N},$$

then (Exercise!) $a_k \to 1$ as $k \to \infty$. Hence $\sum a_k$ is divergent, by the Non-null test.

Example ($\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$)

Let $a_n = 2^{-n}$, for $n \ge 1$. The the k-th partial sum

$$s_k = 2^{-1} + 2^{-2} + \cdots + 2^{-k}, \qquad k \ge 1,$$

satisfies

$$2^{-1}s_k = 2^{-2} + 2^{-3} + \cdots + 2^{-(k+1)}.$$

Subtracting the second equation from the first, we obtain

$$2^{-1}s_k = 2^{-1} - 2^{-(k+1)},$$

whence

$$s_k = 1 - 2^{-k}.$$

Thus $s_k \to 1$ as $k \to \infty$.

Exercise: Prove that

$$\sum_{n=N}^{N+P} 2^{-n} = 2^{-(N-1)} \left(1 - 2^{-(P+1)}\right) \le 2^{-(N-1)},$$

for any positive integers N and P.
Of course, this is an example of a geometric series.

Theorem (Sum of Geometric Series)

If $|w| < 1$, *then*

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}.$$

Proof.

Let

$$s_n = 1 + w + w^2 + \cdots + w^{n-2} + w^{n-1}.$$

Then

$$ws_n = w + w^2 + \cdots + w^{n-1} + w^n.$$

Subtracting these equations, we find

$$(1-w)\, s_n = 1 - w^n \quad \text{or} \quad s_n = \frac{1-w^n}{1-w}.$$

Hence $\left|\frac{1}{1-w} - s_n\right| = \frac{|w|^n}{|1-w|} \to 0$, as $n \to \infty$, since $|w| < 1$. □

Example

Let

$$b_n = \frac{10^{10} + \sin(25n)}{2^n}, \qquad n \ge 1.$$

Then the series $\sum_{n=1}^{\infty} b_n$ is convergent. The key point is comparison with the convergent geometric series. Specifically, we have

$$\sum_{n=N}^{N+P} |b_n| \le C \sum_{n=N}^{N+P} 2^{-n} \le C2^{-(N-1)}.$$

where $C = 1 + 10^{10}$. Thus, given any $\epsilon > 0$, there exists N such that $C2^{-(N-1)} < \epsilon$, and the series is convergent.

This is another variant of the Comparison Test:

Theorem (Comparison Test 3)

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series and suppose that there exists a positive integer N and a positive constant K for which $|b_n| \le K|a_n|$ for $n \ge N$. Then $\sum_{n=1}^{\infty} b_n$ is also an absolutely convergent series

Proof.

Given any $\epsilon >$, choose a positive integer N so large that

$$\sum_{n \ge N} |a_n| < \frac{\epsilon}{K}.$$

Then

$$\sum_{n \ge N} |b_n| < \epsilon.$$

□

Comparing series to geometric series

Theorem (Ratio Test 1)

Let (a_k) be a sequence of non-zero numbers for which the ratio of successive terms is convergent, i.e.

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = r.$$

1. *If $r < 1$, then $\sum a_k$ is absolutely convergent.*
2. *If $r > 1$, then $\sum a_k$ is divergent.*

In other words, if the tail of the sequence ultimately looks like an exponentially decreasing sequence, then the series is absolutely convergent, but if the tail is ultimately exponentially increasing, then the series is divergent.

Proof.

1. For any $R \in (r, 1)$, there exists $N \in \mathbb{N}$ for which
$$|a_{k+1}| < R|a_k|, \qquad \text{for } k \geq N.$$
Hence for $n \geq N$ we have
$$\sum_{k=n}^{n+p} |a_k| < \left(1 + R + R^2 + \cdots + R^{p-1}\right) R^{n-N}|a_N| \leq \frac{R^{n-N}}{1-R}|a_N|$$
and the RHS tends to zero, as $n \to \infty$.
2. If $r > 1$, then for any $R \in (1, r)$ there exists $N \in \mathbb{N}$ for which
$$|a_{k+1}| \geq R|a_k|, \qquad \text{for all } k \geq N.$$
Thus (a_k) fails the Non-null test, since these increase geometrically.

□

We don't actually need the ratios a_{k+1}/a_k to tend to a limit.

Theorem (Ratio test 2)

1. *Let (a_k) be a sequence of non-zero numbers for which the ratio of successive terms satisfies*

$$\left|\frac{a_{k+1}}{a_k}\right| \le r < 1,$$

for all sufficiently large $k \in \mathbb{N}$. Then $\sum a_k$ is absolutely convergent.

2. *If $r > 1$ and there is a subsequence n_k for which*

$$\left|\frac{a_{n_{k+1}}}{a_{n_k}}\right| \ge r,$$

for all $k \in \mathbb{N}$, then $\sum a_k$ is divergent.

Proof.

1. Exercise: Modify the proof of Ratio Test 1 to prove this.
2. If there's a subsequence whose absolute values grow exponentially, then it cannot satisfy the Non-null Test, i.e. we don't have $a_n \to 0$.

□

Example (The Ratio Test only works for geometric-like series.)

We have already seen that the series

$$\sum_{k=0}^{\infty} 1/k^2$$

is convergent. Unfortunately, here the Ratio Test tells us nothing, since, if $a_k = 1/k^2$, then

$$\frac{a_{k+1}}{a_k} = \frac{k^2}{(k+1)^2} = \frac{1}{(1+1/k)^2} \to 1,$$

as $k \to \infty$. The Ratio Test is really just a simple way to test whether a series is geometric-like, in the sense that the absolute values of its terms are, at worst, exponentially decreasing. It can tell us nothing about $\sum k^{-2}$, for which the terms only decrease algebraically.

Example (Exponential Taylor series is absolutely convergent)

We shall prove that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is absolutely convergent for all $x \in \mathbb{R}$. Indeed, setting $a_n = x^n/n!$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \to 0,$$

as $n \to \infty$, so the series is absolutely convergent by the Ratio test.

For a geometric progression $a_n = r^n$, the n-th root is constant: $a_n^{1/n} = r$. If the n-th root is approximately constant for all sufficiently large n, then the series is almost a geometric series, as we see in the next example.

Example

Let $a_n = n\,(0.9)^n$, for $n \ge 0$. Now $n^{1/n} \to 1$, as $n \to \infty$, which implies $a_n^{1/n} \to 0.9$. Hence there exists a positive integer N for which $a_n^{1/n} < 0.91$ for all $n \ge N$, which implies that $a_n < 0.91^n$ for $n \ge N$. Thus the series $\sum a_n$ is convergent, by comparison with the geometric series $\sum 0.91^n$.

This can be generalized to any sequence for which $a_n^{1/n} \to r$ and $r \in (0,1)$, and this is called the *Root test*:

Theorem

(The Root Test) *Let a_n be any positive sequence for which $\lim_{n\to\infty} a_n^{1/n} = r$. If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent. However, if $r > 1$, then the series is divergent.*

Proof: Given any $\epsilon > 0$, there exists a positive integer N for which

$$\left| a_n^{1/n} - r \right| < \epsilon, \qquad n \geq N,$$

i.e.

$$r - \epsilon < a_n^{1/n} < r + \epsilon, \qquad n \geq N.$$

Hence, if $r < 1$, then for any $s \in (r, 1)$, there exists N such that

$$a_n^{1/n} < s, \qquad n \geq N,$$

since we can choose any ϵ for which $r + \epsilon < s$. But then we have

$$a_n < s^n, \qquad \geq N,$$

and the series is convergent, by the comparison principle.

However, if $r > 1$, then for any $s \in (1, r)$, there exists N for which

$$a_n^{1/n} > s, \qquad n \geq N.$$

Hence $a_n > s^n$, for $n \geq N$, and this is increasing exponentially. Thus the series is divergent.

The Root Test is useful when a series is fairly "close" to a geometric series, as the following example will show.

Exercise: Prove that $\sum_{n=1}^{\infty} n^2 2^{-n}$ is convergent.

Exercise: We don't need the n-th root to actually converge to compare the series with a geometric series. You should not find it too difficult to prove the following slightly stronger statements:

1. Suppose that there is a constant $r \in (0,1)$ such that $a_n^{1/n} < r$ for all sufficiently large n. Prove that $\sum a_n$ is convergent.
2. Suppose that there exists a constant $r > 1$ for which $a_n^{1/n} > r$ infinitely often, i.e. for infinitely many values of n. Prove that $\sum a_n$ is divergent.

Example (Root Test beats Ratio Test)

The Ratio Test is often easier to apply than the Root Test, but the Root Test is more powerful. If we consider the convergent (Why?) series

$$1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2} + \cdots$$

i.e.

$$a_{2k} = \frac{1}{2^k} \quad \text{and} \quad a_{2k+1} = \frac{1}{2^k}, \quad k = 0, 1, 2, \ldots,$$

then

$$\frac{a_{2k+1}}{a_{2k}} = 1, \qquad \text{for } k \geq 0,$$

so the Ratio Test is inconclusive.

Exercise: Show that

$$\lim_{n\to\infty} a_n^{1/n} = \frac{1}{2}.$$

Example

The exponential series

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is absolutely convergent for all $x \in \mathbb{R}$ (or, indeed $x \in \mathbb{C}$). Given any $x \in \mathbb{R}$, choose any $r \in (0,1)$. Then there exists a positive integer N for which $(|x|/N) < r$. Hence, for any $k \geq N$, we have

$$\frac{|x|^k}{k!} = \left(\frac{|x|^N}{N!}\right)\left(\frac{|x|}{N+1}\right)\cdots\left(\frac{|x|}{k}\right) < \left(\frac{|x|^N}{N!}\right) r^{k-N}.$$

Thus it's absolutely convergent by comparison with a geometric series.

Theorem (The Alternating Series Test)

Let (a_k) be a decreasing sequence of positive numbers that tends to zero, i.e. $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and $a_k \to 0$, as $k \to \infty$. Then

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

is convergent. Further, the partial sums

$$S_n = \sum_{k=1}^{n} (-1)^{k-1} a_k$$

satisfy $|S_n - S| \leq |a_{n+1}|$, for all $n \in \mathbb{N}$, where $S = \lim_{n\to\infty} S_n$.

Proof.

First note that

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}),$$

and each bracketed term is positive, so (S_{2m}) is an increasing sequence. Further,

$$S_{2m} = a_1 + (-a_2 + a_3) + (-a_4 + a_5) \cdots + (-a_{2m-2} + a_{2m-1}) - a_{2m} < a_1$$

because each bracketed term is negative. Thus (S_{2m}) is a bounded increasing sequence, and therefore convergent, with limit S say. Now $S_{2m+1} = S_{2m} + a_{2m+1}$ and $a_{2m+1} \to 0$, so $\lim S_{2m+1} = S$ too. Hence the series is convergent. The last part is an exercise.

□

The Harmonic Series

The *harmonic series* is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

and here is the fundamental result.

Theorem

$\sum_{n=1}^{\infty} 1/n$ *is divergent.*

Exercise: Use the Alternating Series test to prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is convergent. How quickly does it converge?

Proof.

Let

$$T_k = \{n \in \mathbb{Z}_+ : 2^{k-1} < n \le 2^k\},$$

for $k = 1, 2, \ldots$. Thus $T_1 = \{2\}$, $T_2 = \{3, 4\}$, $T_3 = \{5, 6, 7, 8\}$, etc. We see that the number of elements $|T_k| = 2^{k-1}$. Further, if $n \in T_k$, then $n^{-1} \ge 2^{-k}$. Hence

$$\sum_{n \in T_k} n^{-1} \ge |T_k| \min\{n^{-1} : n \in T_k\} = 2^{k-1} 2^{-k} = \frac{1}{2},$$

for every positive integer k. Therefore

$$\begin{aligned}\sum_{n=1}^{2^M} n^{-1} &= 1 + \sum_{n \in T_1} n^{-1} + \sum_{n \in T_2} n^{-1} + \cdots \sum_{n \in T_M} n^{-1} \\ &\ge 1 + \frac{M}{2}\end{aligned}$$

and this is not bounded above, since M can be arbitrarily large. □

The harmonic series is divergent, but rather slowly so. Later, we shall prove that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^{-1}}{\log n} = 1.$$

Example

Here is an entirely different proof of the divergence of the harmonic series, this time proceeding by contradiction. Let

$$S_m = \sum_{n=1}^{m} n^{-1}$$

and suppose S_m is bounded above for all m. In other words, suppose that $\alpha = \sup_m S_m$ exists and is finite, so that $\lim_{m\to\infty} S_m = \alpha$. There there exists a positive integer M for which $S_M > \alpha - 1/2$, which implies

$$\begin{aligned} \alpha > S_{4M} &= S_M + \sum_{n=M+1}^{4M} n^{-1} \\ &> S_M + \frac{3M}{4M} = S_M + \frac{3}{4} \\ &> \alpha - \frac{1}{2} + \frac{3}{4} = \alpha + \frac{1}{4}. \end{aligned}$$

The proof technique used to demonstrate the divergence of the harmonic series can also be used to show the convergence of the series

$$\sum_{n=1}^{\infty} n^{-2}. \tag{0.1}$$

Specifically, we define the partition $T_1, T_2, \ldots$ as above and note that the upper bound $n \in T_k$ implies $n > 2^{k-1}$, whence $n^{-2} < 2^{2-2k}$. Hence

$$\sum_{n \in T_k} n^{-2} < 2^{k-1} 2^{2-2k} = 2^{1-k},$$

and

$$\begin{aligned}\sum_{n=1}^{2^M} n^{-2} &= 1 + \sum_{n \in T_1} n^{-2} + \sum_{n \in T_2} n^{-2} + \cdots \sum_{n \in T_M} n^{-2} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{M-1}} \\ &< 3.\end{aligned}$$

This is a fairly rough and ready bound, but the key is that we have shown that it is finite. An entirely different approach shows that

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6},$$

a result usually attributed to Euler.

Non-examinable fun: Big-Oh and little-oh notation

The following notation often shortens comparison arguments between terms of sequences and series.

Definition

Let (y_n) be a sequence of positive numbers.

1. *If there is a constant K for which the sequence (x_n) satisfies $|x_n| \le K y_n$, for all n, then we write $x_n = O(y_n)$, which we state as "x_n is big-Oh of y_n".*
2. *If $\lim_{n\to\infty} x_n/y_n = 0$, then we write $x_n = o(y_n)$, which we state as "x_n is little-oh of y_n".*
3. *If $\lim_{n\to\infty} x_n/y_n = 1$, then we write $x_n \sim y_n$, which we state as "x_n is asymptotically equal to y_n".*

As if so often the case, the best way to understand the power of this new notation is via example.

Example

1. If $x_n = 7n^2 - 300n + 1000$, then $x_n = O(n^2)$, $x_n \sim 7n^2$ and $x_n = o(n^3)$; in fact, $x_n = o(n^k)$ for any $k > 2$.
2. If $y_n = 1$, for all n, then $x_n = O(y_n)$ means that the sequence (x_n) is bounded. We usually just write $x_n = O(1)$. Similarly, $x_n = o(1)$ means that $x_n \to 0$ as $n \to \infty$.
3. If $x_n = (n+1)^2$ and $y_n = n^2$, then $x_n \sim y_n$, but it is *not* true that $x_n - y_n \to 0$, since $x_n - y_n = 2n + 1 \to \infty$.
4. Let
$$x_n = \frac{7n^2 - 1000n + 10^5}{n^k},$$
where k is a constant. Then $x_n = O(1/n^{k-2})$, so $\sum x_n$ is convergent for $k > 3$. Further, $x_n \sim 7/n^{k-2}$, so $\sum x_n$ diverges if $k \leq 3$.

We have defined O, o and $\sim$ for sequences, but they are also adaptable to functions, with very minor changes in formal definition.

Example

1. As $x \to \infty$, $x^{1000} = o(e^x)$ and $\cos x = O(1)$.
2. As $x \to 0+$ (i.e. tending to zero through positive values only), $x^2 = o(x)$, $x^3 = o(x^2)$, $e^{-1/x} = o(1)$, $\sin x = O(x)$ and $\sin x \sim x$.
3. As $x \to 0$, $\sin(1/x) = O(1)$ and $1 - \cos x \sim x^2/2$.