Real Analysis 4: Continuity

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You can download these slides and the lecture videos from my office server

http://econ109.econ.bbk.ac.uk/brad/analysis/

Recommended book: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.

Example (A once pathological function)

Consider the following function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q}. \end{cases}$$

In any interval (a, b),

$$\sup\{f(x) : a < x < b\} = 1 \quad \text{and} \quad \inf\{f(x) : a < x < b\} = 0.$$

What would the graph of $f(x)$ look like? What, if anything, is

$$\int_0^1 f(x)\,dx?$$

Such functions were disturbing in the earlier days of analysis, hence the term "pathological". Continuity is one way to avoid them.

A continuous function has no "jumps".

Definition

1. We say $f : [a, b] \to \mathbb{R}$ is *continuous* at $c \in (a, b)$ if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.
2. We say that f is continuous at a if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $x \in (a, a + \delta)$ implies $|f(x) - f(a)| < \epsilon$.
3. We say that f is continuous at b if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $x \in (b - \delta, b)$ implies $|f(x) - f(b)| < \epsilon$.

Sequence definition of continuity: If $x_n \to c$, then $f(x_n) \to f(c)$.

Example (x^2 is continuous)

$f(x) = x^2$ is continuous on $\mathbb{R}$, i.e. continuous at every point $c \in \mathbb{R}$. To see this, first note that

$$f(c+h) - f(c) = (c+h)^2 - c^2 = 2ch + h^2,$$

and we want to prove that this is small for sufficiently small $|h|$. If we choose any $R \in (0, 1)$, then $|h| \le R$ implies

$$|f(c+h) - f(c)| = \left|2ch + h^2\right| \le 2|c|R + R^2 \le (2|c| + 1)\, R,$$

since $R^2 < R$ for $R \in (0, 1)$. Thus, given any $\epsilon > 0$, if we pick $\delta < \epsilon / (2|c| + 1)$, then $|h| < \delta$ implies that

$$|f(c+h) - f(c)| \le (2|c| + 1)\, \delta < \epsilon.$$

Example (x^n is continuous)

We could mimic the proof of continuity of x^2 to prove that $f(x) = x^n$ is continuous. The crucial point is that, for $|h| < 1$,

$$\begin{aligned} |f(c+h) - f(c)| &\le \sum_{k=1}^{n} \binom{n}{k} |h|^k |c|^{n-k} \\ &\le |h| \sum_{k=1}^{n} \binom{n}{k} |c|^{n-k}. \end{aligned}$$

Exercise: Complete the proof.

Example (A discontinuous function)

Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q}. \end{cases}$$

Then in every open interval (a, b) we have

$$\sup\{f(x) : a < x < b\} = 1 \quad \text{and} \quad \inf\{f(x) : a < x < b\} = 0.$$

Hence f is nowhere continuous.

Theorem (Sums of continous functions are continuous.)

If f_1 and f_2 are continuous functions on $[a, b]$, then so is $g = f_1 + f_2$.

Proof.

We have

$$\begin{aligned} |g(x) - g(c)| &= |(f_1(x) - f_1(c)) + (f_2(x) - f_2(c))| \\ &\le |f_1(x) - f_1(c)| + |f_2(x) - f_2(c)| . \end{aligned}$$

Now choose any $\epsilon > 0$. By continuity of f_1 and f_2, we know that there exists $\delta_k > 0$ such that $|x - c| < \delta_k$ implies $|f_k(x) - f_k(c)| < \epsilon/2$. Hence, if $|x - c| < \delta := \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned} |g(x) - g(c)| &= |(f_1(x) - f_1(c)) + (f_2(x) - f_2(c))| \\ &\le |f_1(x) - f_1(c)| + |f_2(x) - f_2(c)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Theorem (Products of continous functions are continuous.)

If f_1 and f_2 are continuous functions on $[a, b]$, then so is $g(x) = f_1(x)f_2(x)$, for $x \in [a, b]$.

Proof.

Given any $\epsilon > 0$, there exists $\delta_k > 0$ such that $|x - c| < \delta_k$ implies $|f_k(x) - f_k(c)| < \epsilon$. Hence if $\delta = \min\{\delta_1, \delta_2\}$, then $|x - c| < \delta$ implies $|f_k(x) - f_k(c)| < \epsilon$ and $|f_k(x)| < |f_k(c)| + \epsilon$ for $|x - c| < \delta$. Therefore

$$
\begin{aligned}
&|f_1(x)f_2(x) - f_1(c)f_2(c)| \\
&= |f_1(x)f_2(x) - f_1(c)f_2(x) + f_1(c)f_2(x) - f_1(c)f_2(c)| \\
&\le |f_1(x)f_2(x) - f_1(c)f_2(x)| + |f_1(c)f_2(x) - f_1(c)f_2(c)| \\
&= |f_2(x)|\,|f_1(x) - f_1(c)| + |f_1(c)|\,|f_1(x) - f_1(x)| \\
&< (|f_2(c)| + \epsilon)\,\epsilon + |f_1(c)|\,\epsilon.
\end{aligned}
$$

□

Lemma

Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(c) > 0$ for some point $a < c < b$. Then exists an open $(c - \delta, c + \delta) \subset [a, b]$ such that $f(x) > f(c)/2$ for $|x - c| < \delta$.

Proof.

Let $\epsilon = f(c)/2$. By continuity, there exists an an open interval $(c - \delta, c + \delta)$ in which

$$|f(x) - f(c)| < \epsilon = \frac{f(c)}{2}.$$

In other words,

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2},$$

which implies that $\frac{f(c)}{2} < f(x) < \frac{3}{2} f(c)$, for $x \in (c - \delta, c + \delta)$. □

Optional extra proof of Bolzano–Weierstrass

Recall that a real sequence (a_n) is just a function $f\colon \mathbb{N} \to \mathbb{R}$, i.e. $a_n = f(n)$.

Definition

We shall call $m \in \mathbb{N}$ *a* **peak number** *if* $a_m \geq a_k$ *for all* $k \geq m$.

Theorem (Bolzano–Weierstrass with a different proof)

Any bounded sequence (a_n) *of real numbers contains a monotonic, and therefore convergent, subsequence.*

Optional extra proof of Bolzano–Weierstrass

Bolzano–Weierstrass via hidden monotonic sequences.

Either: there are infinitely many peak numbers $p(1) < p(2) < p(3) < \cdots$, then $a_{p(k)} \geq a_{p(k+1)}$, i.e. $a_{p(1)}, a_{p(2)}, \ldots$ is a bounded decreasing subsequence.

Or: there are only finitely many peak numbers. Let M be the greatest peak number. For every $n > M$, n is not a peak number, so there must exist a least $g(n) > n$ with $a_{g(n)} > a_n$.

Define $q(1) = M + 1$ and $q(k+1) = g(q(k))$. Then $q(k) < q(k+1)$ and $a_{q(k)} < a_{q(k+1)}$ for all k, so $(a_{q(k)})$ is a bounded increasing subsequence.

Finally, a bounded monotonic subsequence is convergent. □

Lemma (Continuous functions are locally bounded)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and choose any point $c \in (a, b)$ for which $f(c) \neq 0$. Then exists an open interval $(c - \delta, c + \delta) \subset [a, b]$ such that

$$|f(x)| \leq 2\,|f(c)|\,,$$

for $|x - c| < \delta$.

Proof.

By continuity, there exists an open interval $(c - \delta, c + \delta)$ in which

$$|f(x) - f(c)| < |f(c)|.$$

In other words,

$$-|f(c)| < f(x) - f(c) < |f(c)|,$$

which implies $|f(x)| < 2|f(c)|$. □

Theorem

Let $f : [a,b] \to \mathbb{R}$ be any unbounded function. Then f is not continuous everywhere in $[a,b]$, i.e. there exists a point at which f is discontinuous.

Proof.

Without loss of generality, f is not bounded above. Thus there exists a sequence (x_n) for which $f(x_n) > n$. By Bolzano–Weierstrass, there is a convergent subsequence (x_{n_k}), with limit $c \in [a,b]$. Thus, given any $\delta > 0$, there exists $N \in \mathbb{N}$ for which $|x_{n_k} - c| < \delta$, for $n_k \geq N$, and $f(x_{n_k}) > |f(c)|$. Hence, using the Triangle Inequality $|A - B| \geq \Big| |A| - |B| \Big|$,

$$|f(x_{n_k}) - f(c)| \geq \Big| |f(x_{n_k})| - |f(c)| \Big| > n_k - |f(c)|.$$

Thus f is not continuous at c. □

Hence a continuous function on $[a,b]$ is bounded.

Exercise: Is the theorem true in (a,b)?

Theorem (Intermediate Value Theorem 1)

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < f(b)$, then, for every $y \in (f(a), f(b))$, there exists at least one point $c \in (a, b)$ for which $f(c) = y$.

Proof.

Let

$$S = \{x \in [a, b] : f(x) < y\}.$$

Then $a \in S$, so it's non-empty and contained in the bounded interval $[a, b]$. Hence $c = \sup S$ exists. If $f(c) > y$, then there exists $\delta > 0$ such that $f(x) > y$ for $|x - c| \le \delta$. However, if this is so, then $c - \delta$ is an upper bound for S, contradicting the definition of c. Similarly, if $f(c) < y$, then there exists $\delta > 0$ such that $f(x) < y$ for $|x - c| \le \delta$, again contradicting the definition of c. The only remaining possibility is $f(y) = c$. □

Exercise: What happens if $f(a) > f(b)$? **Hint:** Consider $-f$.

Theorem

(IVT2) *If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) > 0 > f(b)$, then there exists at least one point $c \in (a, b)$ for which $f(c) = 0$.*

CONSTRUCTIVE PROOF:

Let $a_0 = a$, $b_0 = b$ and let $L = b_0 - a_0$. Thus $f(a_0) > 0 > f(b_0)$ and our interval $[a_0, b_0]$ has length L.

Either $f(\frac{a_0+b_0}{2}) = 0$, in which case **STOP**

or $f(\frac{a_0+b_0}{2}) \neq 0$.

If $f(\frac{a_0+b_0}{2}) > 0$, then let $a_1 = (a_0 + b_0)/2$, $b_1 = b_0$,

else let $a_1 = a_0$, $b_1 = (a_0 + b_0)/2$.

We now have a new interval $[a_1, b_1]$ of length $L/2$ for which $f(a_1) > 0 > f(b_1)$. We can then repeat the construction, each time obtaining an interval $[a_k, b_k]$ of length $L/2^k$ for which $f(a_k) > 0 > f(b_k)$.

Specifically, at the k-th stage of our algorithm, for $k \geq 1$, we repeat this procedure:

Either $f(\frac{a_k+b_k}{2}) = 0$, in which case **STOP**

or $f(\frac{a_k+b_k}{2}) \neq 0$.

If $f(\frac{a_k+b_k}{2}) > 0$, then let $a_{k+1} = (a_k + b_k)/2$, $b_{k+1} = b_k$,

else let $a_{k+1} = a_k$, $b_{k+1} = (a_k + b_k)/2$.

To conclude, we have generated two sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ with the following properties:

1. $\{a_k\}$ is an increasing sequence,
2. $\{b_k\}$ is a decreasing sequence,
3. each a_k is bounded above by $b_1, b_2, \ldots$,
4. each b_ℓ is bounded below by $a_1, a_2, \ldots$, and
5. $b_k - a_k = L/2^k$.

Since bounded monotonic sequences are convergent, we deduce the existence of $\alpha = \lim_{k\to\infty} a_k$ and $\beta = \lim_{k\to\infty} b_k$. Further $\alpha \le \beta$, $f(\alpha) \ge 0 \ge f(\beta)$ and $[\alpha, \beta] \subset [a_k, b_k]$, for every integer k (Why?). Hence $\beta - \alpha \le L/2^k$, for every k, which implies $\alpha = \beta$. We then conclude that $f(\alpha) \ge 0$ and $f(\alpha) \le 0$, since α is the limit of the $\{a_k\}$ and the $\{b_k\}$, which implies that $f(\alpha) = 0$. Thus $c = \alpha$ is our desired root. $\square$

Example

A continuous function can have arbitrarily many roots in a closed interval: consider $a = 0$, $b = \pi$, $f(x) = \cos nx$, where n is an odd positive integer.

Exercise

Give an example of a continuous function with infinitely many roots in $(0, 1]$. **Hint:** *First find an example of a continuous function with infinitely many roots in* $[1, \infty)$.

The next result will again use a bisection argument. I shall use the notation

$$\sup_I f = \sup\{f(x) : x \in I\},$$

and

$$\inf_I f = \inf\{f(x) : x \in I\},$$

for any interval I.

Key point: At least one of $\sup_{[a,(a+b)/2]} f$ and $\sup_{[(a+b)/2,b]} f$ must be equal to $\sup_{[a,b]} f$.

The proof that $\inf_I f$ is attained is essentially identical.

Theorem

If $f : [a, b] \to \mathbb{R}$ is a continuous function, then f attains its bounds.

CONSTRUCTIVE PROOF:

We begin the bisection argument as before:

If $\sup_{[a_0,(a_0+b_0)/2]} f = \sup_{[a_0,b_0]} f$, or if

$$\sup_{[a_0,(a_0+b_0)/2]} f = \sup_{[(a_0+b_0)/2,b_0]} f,$$

then let $a_1 = a_0,\ b_1 = (a_0 + b_0)/2$,
else let $a_1 = (a_0 + b_0)/2,\ b_1 = b_0$.
We now have a new interval $[a_1, b_1]$ of length $L/2$ for which $\sup_{[a_1,b_1]} f = \sup_{[a_0,b_0]} f$.

As in the bisection proof of the Intermediate Value Theorem, we now repeat the construction, each time obtaining an interval $[a_k, b_k]$ of length $L/2^k$ on which $\sup_{[a_k,b_k]} f = \sup_{[a_0,b_0]} f$.
To conclude, we have generated two sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ with the following properties:

1. $\{a_k\}$ is an increasing sequence,
2. $\{b_k\}$ is a decreasing sequence,
3. each a_k is bounded above by $b_1, b_2, \ldots$,
4. each b_ℓ is bounded below by $a_1, a_2, \ldots$, and
5. $b_k - a_k = L/2^k$.

Since bounded monotonic sequences are convergent, we deduce the existence of $\alpha = \lim_{k\to\infty} a_k$ and $\beta = \lim_{k\to\infty} b_k$. Further $\alpha \le \beta$, $f(\alpha) \ge 0 \ge f(\beta)$ and $[\alpha, \beta] \subset [a_k, b_k]$, for every integer k (Why?). Hence $\beta - \alpha \le L/2^k$, for every k, which implies $\alpha = \beta$. Thus $\sup_{[a_k,b_k]} f = \sup_{[a,b]} f$, $b_k - a_k = L/2^k$ and $\alpha \in [a_k, b_k]$, for all k. However, f is continuous at α. Hence, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|x - \alpha| \le \delta$ implies $|f(x) - f(\alpha)| \le \epsilon$. Finally, $[a_k, b_k] \subset [\alpha - \delta, \alpha + \delta]$, for all sufficiently large k, which implies that $f(\alpha) = \sup_{[a,b]} f$. □

Example (Non-uniformly continuous function)

Consider the function $f(x) = 1/x$ for $0 < x \le 1$. Then

$$\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy}.$$

If $0 < r < 1$ and we let $x = r$, $y = 2r$, then

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{2r}$$

which tends to infinity as $r \to 0$.

Definition (Uniform continuity)

Let $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}$. We say that f is **uniformly continuous** *on E iff, for any $\epsilon > 0$, there exists $\delta > 0$ for which*

$$|x - y| < \delta \quad \textit{implies} \quad |f(x) - f(y)| < \epsilon$$

for all $x, y \in E$.

In symbols ($\forall$ = "for all", $\exists$ = 'there exists"), uniform continuity requires

$$\forall \epsilon \; \exists \delta > 0 \; \forall x, y \in E : \; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Compare this with continuity:

$$\forall x \in E \; \forall \epsilon \; \exists \delta > 0 \; \forall y \in E : \; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Key difference: δ only depends on ϵ for uniform continuity, while it's a function of both ϵ and x for continuity.

Theorem

If $f : [a, b] \to \mathbb{R}$ is not uniformly continuous, then it's discontinuous at some point in $[a, b]$

Proof.

If f were NOT uniformly continuous, then

$$\exists \epsilon > 0 \; \forall \delta > 0 \; \exists x, y \in [a, b] : |x - y| < \delta \;\text{ AND }\; |f(x) - f(y)| > \epsilon.$$

In other words, there exists $\epsilon > 0$ and two sequences (x_n) and (y_n) in $[a, b]$ for which

$$|f(x_n) - f(y_n)| > \epsilon \;\text{ AND }\; |x_n - y_n| < \frac{1}{n}.$$

By Bolzano–Weierstrass, these sequences have convergent subsequences (x_{n_k}) and (y_{n_k}) with a common limit $c \in [a, b]$ (Why?). Hence f is not continuous at c, since

$$0 < \epsilon < |f(x_{n_k}) - f(y_{n_k})| \le |f(x_{n_k}) - f(c)| + |f(c) - f(y_{n_k})|.$$