Real Analysis 5.5: Complex Numbers

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You can download these slides and the lecture videos from my office server

`http://econ109.econ.bbk.ac.uk/brad/analysis/`

Recommended books: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.
J. C. Burkill (1978), "A First Course in Mathematical Analysis", Cambridge University Press.

Definition

Let

$$\mathcal{C} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

and

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \mathbb{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\mathcal{C} = \{a\mathbb{1} + b\mathbb{I} : a, b \in \mathbb{R}\}$$

and $\mathbb{I}$ *is the matrix for clockwise rotation through* $\pi/2$ *radians.*

We call $\mathcal{C}$ the set of **rotation-enlargement** matrices, for reasons that will become clear later.

Theorem (1 and J)

Every rotation–enlargement matrix can be uniquely written in the form

$$a\mathbb{1} + b\mathbb{I}.$$

Further, the relation

$$\mathbb{I}^2 = -\mathbb{1}$$

implies the multiplication rule

$$(a_1\mathbb{1} + b_1\mathbb{I})\,(a_2\mathbb{1} + b_2\mathbb{I}) = (a_1 a_2 - b_1 b_2)\,\mathbb{1} + (a_1 b_2 + b_1 a_2)\,\mathbb{I}$$

and we observe that $ZW = WZ$ *for any* $Z, W \in \mathcal{C}$ *because* $\mathbb{1}$ *commutes with every matrix.*

Theorem

$\mathcal{C}$ contains a copy of $\mathbb{R}$:

$$\{a\mathbb{1} : a \in \mathbb{R}\}.$$

The subset

$$\mathrm{IM} = \{b\mathbb{I} : b \in \mathbb{R}\}$$

is the set of all real multiples of the rotation matrix $\mathbb{I}$.

Example

If $Z = a\mathbb{1} + b\mathbb{I}$, then

$$Z(a\mathbb{1} - b\mathbb{I}) = (a^2 + b^2)\mathbb{1}.$$

Definition

We write $Z^* = a\mathbb{1} - b\mathbb{I}$, *so that*

$$ZZ^* = Z^*Z = (a^2 + b^2)\mathbb{1}.$$

The **modulus** *of* Z *is defined by* $|Z| = \sqrt{ZZ^*} = \sqrt{a^2 + b^2}$.

Note that Z^* is also the matrix transpose of Z.

Theorem

If $Z = a\mathbb{1} + b\mathbb{I}$ *and* $|Z| > 0$, *then the matrix* Z *is invertible and*

$$Z^{-1} = \frac{Z^*}{|Z|}.$$

Theorem

If $Z = a\mathbb{1} + b\mathbb{I}$ and $|Z| = 1$, then $Z^{-1} = Z^*$, i.e. $ZZ^* = \mathbb{1}$.

Exercise

The **norm** $\|\mathrm{v}\|$ *of any vector*

$$\mathrm{v} \in \mathbb{R}^2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

is just its Euclidean length, i.e. $\|\mathrm{v}\| = \sqrt{v_1^2 + v_2^2}$. *Show that* $\|Z\mathrm{v}\| = \|\mathrm{v}\|$ *when* $|Z| = 1$.

TRADITIONAL NOTATION: $a + ib \equiv a\mathbb{1} + b\mathbb{I}$.

In geometrical terms, $a\mathbb{1} + b\mathbb{I}$ is a rotation-enlargement matrix: the scale factor is $\sqrt{(}a^2 + b^2)$ and the rotation angle θ satisfies $\tan\theta = b/a$.