Real Analysis 6: The Standard Functions of Analysis

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You can download these slides and the lecture videos from my office server

http://econ109.econ.bbk.ac.uk/brad/analysis/

Recommended books: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.
J. C. Burkill (1978), "A First Course in Mathematical Analysis", Cambridge University Press.

The exponential function is defined by

$$\exp z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \qquad \text{for } z \in \mathbb{C},$$

which is an absolutely convergent series by the ratio test:

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{|z|}{k+1} \to 0,$$

as $k \to \infty$, where $a_k = z^k/k!$. We now need to deduce its other vital properties from this definition.

Notation: It's fine to write e^z or $\exp(z)$, but we shall keep to $\exp(z)$ until we have derived further properties.

Theorem (Can differentiate power series term by term)

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is convergent for $|z| < R$, *then*

$$f'(z) = \sum_{n=1}^{\infty} a_n n z^{n-1}.$$

Proof.

NONEXAMINABLE □

Theorem

$$\frac{d}{dz} \exp z = \exp z.$$

Proof.

Here $a_n = 1/n!$, so that

$$\begin{aligned}
\frac{d}{dz} \exp z &= \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \\
&= \exp z.
\end{aligned}$$

□

Theorem

Choose any $w \in \mathbb{C}$ *and define*

$$f(z) = \exp(w - z)\exp(z), \qquad z \in \mathbb{C}.$$

Then $f'(z) \equiv 0$ *and* $f(z) \equiv f(w)$.

Proof.

$$f'(z) = -\exp(w - z)\exp(z) + \exp(w - z)\exp(z) = 0.$$

Hence f is constant and $f(z) \equiv f(0) = \exp(w)$. □

Theorem

$$\exp(a + b) = \exp(a)\exp(b) \quad \textit{for any } a, b \in \mathbb{C}.$$

Proof.

Let $w = a + b$ and $z = b$ in the previous theorem. □

Alternative.

We already know Taylor's theorem (last lecture) is valid for $f(x) = \exp(x)$:

$$f(x+y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} f^{(k)}(x)$$

or

$$\exp(x+y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} \exp(x) = \exp(x)\exp(y).$$

□

Theorem

For all $z \in \mathbb{C}$

$$\exp(-z)\exp(z) = 1.$$

Thus the exponential function is never zero.

Proof.

$$\exp(-z)\exp(z) = \exp(-z+z) = \exp 0 = 1.$$

□

Theorem

If $x \in \mathbb{R}$, then $\exp x > 0$.

Proof.

$$\exp x = \exp\left(\frac{x}{2} + \frac{x}{2}\right) = \exp(x/2)^2 > 0,$$

because $\exp(x/2)$ is real if x is real. □

Theorem (exp is strictly increasing on $\mathbb{R}$)

If $x, y \in \mathbb{R}$ and $x < y$, then $\exp x < \exp y$.

Proof.

By the Mean Value Theorem, there exists $c \in (x, y)$ for which

$$\exp y - \exp x = (y - x) \exp c > 0.$$

□

Thus $\exp : \mathbb{R} \to (0, \infty)$ is injective: if $\exp x = \exp y$, then $x = y$.

Theorem

Let $e = \exp 1 = 2.71828182845904523536\ldots$. *Then* $\exp n = e^n$, *for any integer* n. *Further,* $\exp(p/q) = e^{p/q}$, *for any integers* p *and* $q \neq 0$. *We* **define**

$$e^z = \exp(z)$$

for all $z \in \mathbb{C}$.

Proof.

$$\exp n = \exp\left(\underbrace{1 + \cdots + 1}_{n}\right) = (\exp 1)^n = e^n, \quad n \in \mathbb{N}.$$

Further, $\exp(-n)\exp(n) = \exp 0 = 1$ implies that $\exp(-n) = 1/\exp n = 1/(e^n)$, which is the definition of e^{-n}, and

$$e^p = \exp p = \exp[(p/q)q] = \exp(p/q)^q.$$

□

Theorem (Non-examinable)

$e = \exp 1$ is irrational.

NON-EXAMINABLE PROOF: Suppose $e = m/n$, where $m, n \in \mathbb{N}$ with no common factors. Now

$$\frac{m}{n} = e = \sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{n} \frac{1}{k!} + \sum_{k=n+1}^{\infty} \frac{1}{k!} \equiv S_1 + S_2.$$

Multiply both sides by $n!$:

$$m(n-1)! = \underbrace{n!S_1}_{\text{integer}} + n!S_2.$$

But

$$
\begin{aligned}
n!S_2 &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\
&< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \\
&= \frac{1}{n},
\end{aligned}
$$

summing the geometric series (exercise). Hence

$$m(n-1)! - n!S_1 = n!S_2.$$

The LHS is an integer, while the RHS is a positive number in $(0, 1)$, which is a contradiction. □

Theorem

For any positive integer n and $x > 0$,

$$\frac{x^n}{\exp x} < \frac{(n+1)!}{x}.$$

Hence $\lim_{n\to\infty} x^n \exp(-x) = 0$.

Proof.

$$\frac{x^n}{\exp x} = \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots} < \frac{x^n}{\frac{x^{n+1}}{(n+1)!}} = \frac{(n+1)!}{x}.$$

□

Exercise

Show that $\exp x \to \infty$ *as* $x \to +\infty$ *and* $\exp x \to 0$ *as* $x \to -\infty$.

Theorem

The exponential function $\exp\colon \mathbb{R} \to (0, \infty)$ *is a bijection.*

Proof.

We already know that it's strictly increasing, so it's an injection. Further, for any $y > 0$, $\exp y > 1 + y$ and, since $\exp x \to 0$ as $x \to -\infty$, there exists $x_0 \in \mathbb{R}$ for which $\exp x_0 < y$. Thus the function $f(x) = \exp x - y$ satisfies $f(y) > 1$ and $f(x_0) < 0$. Hence there must exist $x \in (x_0, y)$ for which $\exp x = y$.

□

Theorem

Let $L\colon (0,\infty) \to \mathbb{R}$ *denote the inverse of the exponential function, i.e.* $L(\exp x) = x$*, for all* $x \in \mathbb{R}$*. Then*

$$\frac{d}{dy} L(y) = \frac{1}{y}, \qquad \text{for } y > 0,$$

and $L(1) = 0$*.*

Proof.

Differentiating $L(\exp x) = x$ using the Chain rule, we have

$$L'(\exp x) \exp x = 1$$

and setting $y = \exp x$ gives $L'(y)y = 1$, Since $\exp(0) = 1$, we must have $L(1) = 0$. □

Of course, $L(y) = \ln y$, the natural logarithm, but we shall keep to $L(y)$ for now.

Theorem

$L(y) \to \infty$ as $y \to \infty$. Further,

$$L(ab) = L(a) + L(b)$$

for any $a, b > 0$.

Proof.

Let $x = L(a)$ and $y = L(b)$. Then $\exp x = a$, $\exp y = b$ and

$$\exp(x + y) = \exp x \exp y,$$

i.e.

$$\exp(L(a) + L(b)) = ab,$$

or

$$L(a) + L(b) = L(ab).$$

□

Definition (x^a for $a \in \mathbb{R}$)

For $x > 0$, define

$$r_a(x) = \exp(aL(x)).$$

Theorem

For $m, n \in \mathbb{N}$

$$r_n(x) = \exp(nL(x)) = \exp(L(x))^n = x^n$$

and

$$r_{m/n}(x) = \exp((m/n)L(x)) = \exp(L(x))^{m/n} = x^{m/n}.$$

Definition

Define $x^a = r_a(x) = \exp(aL(x))$ for $x > 0$ and $a \in \mathbb{R}$.

Example (Usual properties of exponents)

For $x, y > 0$,

$$\begin{aligned}(xy)^a &= \exp(aL(xy)) \\ &= \exp(a[L(x) + L(y)]) \\ &= \exp(aL(x))\exp(aL(y)) \\ &= x^a \cdot y^a.\end{aligned}$$

Further

$$\begin{aligned}x^{a+b} &= \exp((a+b)L(x)) \\ &= \exp(aL(x))\exp(bL(x)) \\ &= x^a \cdot x^b.\end{aligned}$$

Example

We also see that, for $x > 0$, $x^1 = \exp(L(x)) = x$. Further,

$$\begin{aligned}(x^b)^a &= \exp(aL(x^b)) \\ &= \exp(aL(\exp(bL(x)))) \\ &= \exp(abL(x)) \\ &= x^{ab}.\end{aligned}$$

Exercise

Show that $x^{-1} = \exp(-L(x)) = 1/x$, *for* $x > 0$.

Theorem

$$\frac{d}{dx}x^a = ax^{a-1} \qquad \textit{for } x > 0, a \in \mathbb{R}.$$

Proof.

$$\begin{aligned}\frac{d}{dx}\exp(aL(x)) &= \exp(aL(x))(a/x) \\ &= a\exp(aL(x))\exp(-L(x)) \\ &= a\exp((a-1)L(x)) \\ &= ax^{a-1}.\end{aligned}$$

□

Theorem

If $x > 0$, then

$$\frac{d}{da} x^a = L(x) x^a, \qquad a \in \mathbb{R}.$$

Proof.

$$\frac{d}{da} x^a = \frac{d}{da} \exp(aL(x)) = L(x) \exp(aL(x)) = L(x) x^a.$$

□

Example (Nonexaminable: Calculating $L(x)$)

If we choose $x = 1 + a$ and $|a| < 1$, then, setting $y = 1 + s$,

$$L(1+a) = \int_1^{1+a} \frac{1}{y}\, dy = \int_0^a \frac{1}{1+s}\, ds.$$

Now

$$\frac{1}{1+s} = 1 - s + s^2 - s^3 + \cdots$$

and it turns out that we can integrate power series term by term:

$$\begin{aligned} L(1+a) &= \int_0^a \frac{1}{1+s}\, ds \\ &= \int_0^a 1 - s + s^2 - s^3 + \cdots\, ds \\ &= a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \cdots . \end{aligned}$$

Example

If we let $a = 1/2$, then

$$L(1/2) = L(1 - a/2) = -\left(a + \frac{a^2}{2} + \frac{a^3}{3} + \cdots\right)$$

and

$$L(2) \approx \left(a + \frac{a^2}{2} + \frac{a^3}{3} + \cdots + \frac{a^n}{n}\right).$$

If $n = 20$, then we find

$$L(2) \approx 0.6931471370510288$$

which is correct to 5 decimal places.

Definition

We define

$$c(z) = \frac{1}{2}\Big(\exp(iz) + \exp(-iz)\Big)$$

and

$$s(z) = \frac{1}{2i}\Big(\exp(iz) - \exp(-iz)\Big).$$

Of course, $c(z) = \cos z$ and $s(z) = \sin z$, but we shall keep to $c(z)$ and $s(z)$ while deducing their fundamental properties.

Theorem

$$\exp(iz) = c(z) + is(z) \quad \textit{and} \quad \exp(-iz) = c(z) - is(z)$$

and $\exp 0 = 1$ *implies* $c(0) = 1$ *and* $s(0) = 0$. *Further,* $c(z)$ *is an* **even function**, *i.e.* $c(-z) = c(z)$, *while* $s(z)$ *is an* **odd function**, *i.e.* $s(-z) = -s(z)$, *for all* $z \in \mathbb{C}$.

Example

$$c(5i) = (\exp(5) + \exp(-5))/2 \approx 74.2099485$$

and

$$s(5i) = -i(\exp(-5) - \exp(5))/2 \approx 74.2032105777i.$$

Hence

$$c(5i)^2 = \frac{1}{4}\Big(\exp(10) + 2 + \exp(-10)\Big)$$

and

$$s(5i)^2 = -\frac{1}{4}\Big(\exp(10) - 2 + \exp(-10)\Big).$$

Thus

$$c(5i)^2 + s(5i)^2 = 1.$$

Exercise

Check that $c(5i)^2 + s(5i)^2 = 1$ *numerically.*

Theorem

We have

$$c(z)^2 + s(z)^2 = 1$$

for all $z \in \mathbb{C}$.

Proof.

$$\begin{aligned}
&c(z)^2 + s(z)^2 \\
&= \frac{1}{4}\Big(\exp(2iz) + 2 + \exp(-2iz) - \exp(2iz) + 2 - \exp(-2iz)\Big) \\
&= 1.
\end{aligned}$$

□

Theorem

We have $c(0) = 1$, $s(0) = 0$ *and the differential equations*

$$c'(z) = -s(z) \quad \textit{and} \quad s'(z) = c(z).$$

Hence

$$c''(z) + c(z) = s''(z) + s = 0.$$

Proof.

For example, $c(z) = (1/2)(\exp(iz) + \exp(-iz))$ implies

$$c'(z) = (1/2)(i \exp(iz) - i \exp(-iz)) = -s(z).$$

The rest are left as exercises. □

Theorem (The addition formulae for $c(z)$ and $s(z)$)

We have

$$c(z+w) = c(z)c(w) - s(z)s(w)$$

and

$$s(z+w) = s(z)c(w) + c(z)s(w),$$

for any $z, w \in \mathbb{C}$.

Proof.

$$\begin{aligned} c(z+w) + is(z+w) &= \exp(i(z+w)) \\ &= \exp(iz)\exp(iw) \\ &= (c(z)+is(z))(c(w)+is(w)). \end{aligned}$$

Now equate real and imaginary parts. □

Exercise:

$$c(z-w) = c(z)c(w) + s(z)s(w), \qquad \text{for all } z, w \in \mathbb{C}.$$

Theorem

$$c(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

and

$$s(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots,$$

for any $z \in \mathbb{C}$.

Proof.

These are the real and imaginary parts of the absolutely convergent series

$$\exp(iz) = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!},$$

using the fact that $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. Exercise: These series are absolutely convergent for all $z \in \mathbb{C}$, by the Ratio test

. □

Theorem

There exists $w \in (0, 2)$ for which $c(w) = 0$.

Proof.

We know that $c(0) = 1$ and

$$c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

is an alternating series. Hence

$$1 - \frac{x^2}{2!} \le c(x) \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

for $x \in \mathbb{R}$. If $x = 2$, then

$$c(2) \le 1 - \frac{2^2}{2!} + \frac{2^4}{4!} = -1/3.$$

Hence there is a root $w \in (0, 2)$ by the Intermediate Value Theorem. □

Definition

Let $\omega = \inf\{w \in \mathbb{R}_+ : c(w) = 0\}$.

Exercise

Use the addition formulae to prove that

$$c(2z) = 2c(z)^2 - 1 = c(z)^2 - s(z)^2 \quad \text{and} \quad s(2z) = 2s(z)c(z),$$

for any $z \in \mathbb{C}$. *Hence find* $c(\omega/2)$ *and* $s(\omega/2)$.

Theorem ($c(2\omega) = -1$, $c(4\omega) = 1$, $s(2\omega) = s(4\omega) = 0$)

Now $c(\omega) = 0$ *implies that*

$$c(2\omega) = 2c(\omega)^2 - 1 = -1$$

and

$$c(4\omega) = 2c(2\omega)^2 - 1 = 1.$$

Note that $c(z)^2 + s(z)^2 = 1$ *implies that* $s(2\omega) = s(4\omega) = 0$.

Theorem ($c(z)$ and $s(z)$ have period 4ω)

$$c(z + 4\omega) = c(z) \quad \textit{and} \quad s(z + 4\omega) = s(z)$$

for all $z \in \mathbb{C}$.

Proof.

$$c(z + 4\omega) = c(z)c(4\omega) - s(z)s(4\omega) = c(z)$$

and

$$s(z + 4\omega) = s(z)c(4\omega) + c(z)s(4\omega) = s(z).$$

□

Theorem

1. $c(0) = 1$, $c(x) > 0$ *for* $0 \leq x < \omega$.
2. $s'(x) = c(x) > 0$ *for* $0 \leq x < \omega$, *i.e. s is strictly increasing on* $[0, \omega)$. *Thus* $1 = c(\omega)^2 + s(\omega)^2 = s(\omega)^2$ *implies* $s(\omega) = 1$.
3. $c''(x) = -c(x) < 0$ *for* $0 \leq x < \omega$.

Theorem

For all $z \in \mathbb{C}$*:*

1. $c(z-\omega) = c(z)c(\omega) + s(z)s(\omega) = s(z)$.
2. $s(z+\omega) = s(z)c(\omega) + c(z)s(\omega) = c(z)$.
3. $c(z+2\omega) = c(z)c(2\omega) - s(z)s(2\omega) = -c(z)$, *recalling that* $c(2\omega) = 2c(\omega)^2 - 1 = -1$ *and* $s(2\omega) = 2s(\omega)c(\omega) = 0$.

Theorem

We have $c(x) > 0$ *for* $-\omega < x < \omega$ *and* $c(x) = -c(x-2\omega) < 0$ *for* $\omega < x < 3\omega$*. Hence the real zeros of* c *are at* $\{\omega + 2n\omega : n \in \mathbb{Z}\}$ *and the real zeros of* s *are at* $\{2n\omega : n \in \mathbb{Z}\}$.

Theorem

c and s have no zeros in $\mathbb{C} \setminus \mathbb{R}$.

Proof.

If $s(x + iy) = 0$, then $1 = \exp(2i(x + iy) = \exp(2ix)\exp(-2y)$. But $|\exp(2ix)|^2 = c(2x)^2 + s(2x)^2 = 1$. Hence $\exp(-2y) = 1$, which implies $y = 0$. □

Theorem

$$\exp(z + 4i\omega) = \exp z$$

for all $z \in \mathbb{C}$. *In particular,* $\exp(4i\omega) = 1$.

Proof.

$$\exp(z + 4i\omega) = \exp(z)\exp(4i\omega) = \exp(z)(c(4\omega) + is(4\omega)).$$

□

We can stop pretending now: $c(z) \equiv \cos z$, $s(z) \equiv \sin z$ and $\omega = \pi/2$.

Example (Nonexaminable: Viète's Formula)

The addition formula for $\sin x$ gives

$$\begin{aligned}\sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} \\ &= 2^2 \sin \frac{x}{4} \cos \frac{x}{4} \cos \frac{x}{2} \\ &= 2^3 \sin \frac{x}{8} \cos \frac{x}{8} \cos \frac{x}{4} \cos \frac{x}{2}.\end{aligned}$$

A simple induction provides Viète's formula (1593):

$$\frac{\sin x}{2^n \sin \frac{x}{2^n}} = c_n c_{n-1} \cdots c_2 c_1,$$

where $c_k = \cos(x/2^k)$.

Exercise (Nonexaminable)

Prove that

$$\lim_{n\to\infty} \frac{\sin x}{2^n \sin \frac{x}{2^n}} = \frac{\sin x}{x}.$$

Theorem (Nonexaminable)

If $x = \pi/2$, then Viète's formula becomes

$$\frac{2}{\pi} = \lim_{n\to\infty} c_1 c_2 \cdots c_{n-1} c_n,$$

where

$$c_k = \cos \frac{\pi}{2^{k+1}}, \qquad k \in \mathbb{N}.$$

Exercise

Find c_1 and show that $c_{k+1} = \sqrt{(1 + c_k)/2}$.

Theorem (cosh and sinh)

$$\cos(iz) = \frac{1}{2}\left(e^z + e^{-z}\right) \equiv \cosh z.$$

and

$$\sin(iz) = \frac{1}{2i}\left(e^{i(iz)} + e^{-i(iz)}\right) = i \sinh z,$$

where $\cosh z = (e^z + e^{-z})/2$ *and* $\sinh z = (e^z - e^{-z})/2$.

Example

$$1 = \cos^2(iz) + \sin^2(iz) = \cosh^2 z - \sinh^2 z.$$

Example

$$\cosh(2z) = \cos(2iz) = \cos^2(iz) - \sin^2(iz) = \cosh^2 z + \sinh^2 z.$$

Example

$$\sinh(2z) = -i\sin(2iz) = -2i\sin(iz)\cos(iz) = 2\sinh z\cosh z.$$

Example

$$\cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy)$$
$$= \cos x \cosh y - i \sin x \sinh y.$$

Exercise

Show that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

Hence show that

$$\tan(z^*) = [\tan z]^*,$$

where $z = x + iy$ *and* $z^* = x - iy$.