Real Analysis 7: Integration

Brad Baxter
Birkbeck College, University of London

June 13, 2023

You can download these slides and the lecture videos from my office server

http://econ109.econ.bbk.ac.uk/brad/analysis/

Recommended books: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.
J. C. Burkill (1978), "A First Course in Mathematical Analysis", Cambridge University Press.

Example (Integrating x^2)

We have already seen that

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1).$$

Setting $f(x) = x^2$, this implies that

$$\frac{1}{n}\sum_{k=1}^{n} f(k/n) = \frac{1}{6}(1+1/n)(2+1/n) \to \frac{1}{3},$$

as $n \to \infty$. In the 17th century, this limit of a sum was called an **integral** and expressed as

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

$\int$ was a variant of the letter "s" and was the abbreviation of *summa*, the Latin for "sum".

Definition

A **dissection** *D of (a, b) is any finite set*

$$a \equiv x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n \equiv b.$$

We let $L_k = x_k - x_{k-1}$ and define the **norm** *of the dissection by*

$$L(D) = \max_{1 \le k \le n} L_k.$$

If $M_k = \sup_{[x_{k-1}, x_k]} f$ and $m_k = \inf_{[x_{k-1}, x_k]} f$ then the **upper sum** *is*

$$S(f, D) = \sum_{k=1}^{n} M_k L_k$$

and the **lower sum** *is*

$$s(f, D) = \sum_{k=1}^{n} m_k L_k.$$

Example (Integrating x^2)

Let

$$D_n = \left\{ \frac{kt}{n} : k = 0, 1, 2, \ldots, n-1, n \right\}, \text{ so } L_k = L = t/n.$$

Now $f(x) = x^2$ is strictly increasing for $x > 0$, so

$$\begin{aligned} S(f, D_n) &= \sum_{k=1}^{n} (kt/n)^2 (t/n) = \frac{t^3}{n^3} \sum_{k=1}^{n} k^2 \\ &= \frac{t^3}{6}(1 + 1/n)(2 + 1/n) \end{aligned}$$

and

$$s(f, D_n) = S(f, D_n) + Lf(0) - Lf(t) = S(f, D_n) - t^3/n.$$

Then $\lim_{n\to\infty} S(f, D_n) = \lim_{n\to\infty} s(f, D_n) = t^3/3$.

Example (Important: geometric series dissection)

Let $f(x) = 1/x^2$, $x \in \mathbb{R} \setminus \{0\}$ and let $D_n = \{q^k : 0 \le k \le n\}$ where $q = 2^{1/n}$. Then (exercise)

$$S(f, D_n) = \sum_{k=0}^{n-1} \frac{1}{q^{2k}} q^k (q-1) = (q-1) \sum_{k=0}^{n-1} q^{-k}.$$

Hence

$$S(f, D_n) = (q-1) \left(\frac{1 - q^{-n}}{1 - q^{-1}} \right).$$

But $q = 2^{1/n}$, so

$$S(f, D_n) = \frac{1}{2} \left(\frac{q-1}{1 - 1/q} \right) = \frac{q}{2} \to \frac{1}{2}$$

as $n \to \infty$.

Example (Increasing functions are easy)

Let $f\colon [a,b] \to \mathbb{R}$ be a strictly increasing function and let

$$D_n = \left\{ x_k = a + \frac{k}{n}(b-a) : k = 0, 1, 2, \ldots, n-1, n \right\}.$$

Then $L_k = L = (b-a)/n$ and

$$S(f, D_n) = L \sum_{k=1}^{n} f(x_k)$$

while

$$s(f, D_n) = L \sum_{k=0}^{n-1} f(x_k) = S(f, D_n) + Lf(a) - Lf(b).$$

Theorem

Let $M = \sup_{[a,b]} f$, $m = \inf_{[a,b]} f$. *For any dissection* D *of* $[a, b]$, *we have* $M_k \leq M$ *and* $m \leq m_k$, *so that*

$$m(b - a) \leq s(f, D) \leq S(f, D) \leq M(b - a).$$

Hence

$$\{S(f, D) : D \text{ a dissection of } [a, b]\}$$

and

$$\{s(f, D) : D \text{ a dissection of } [a, b]\}$$

are bounded non-empty sets of reals. Let

$$\overline{\int_a^b} f = \inf_D f \text{ and } \underline{\int_a^b} f = \sup_D f.$$

Refining the dissection Let

$$D_1 = \Big\{ a \equiv x_0 < x_1 < \cdots < x_{n-1} < x_n \equiv b \Big\}.$$

Let D_2 be a new dissection formed by adding a new point x'_k in (x_{k-1}, x_k): we say D_2 is a **refinement** of D_1. Now define

$$M'_k \equiv \sup_{[x_{k-1}, x'_k]} f \le M_k, \qquad m'_k \equiv \inf_{[x_{k-1}, x'_k]} f \ge m_k$$

and

$$M''_k \equiv \sup_{[x'_k, x_k]} f \le M_k, \qquad m''_k \equiv \inf_{[x'_k, x_k]} f \ge m_k.$$

Let $L'_k = x'_k - x_{k-1}$ and $L''_k = x_k - x'_k$.

Theorem

$$s(f, D_1) \leq s(f, D_2) \leq S(f, D_2) \leq S(f, D_1).$$

Proof.

$$\begin{aligned} S(f, D_2) &= S(f, D_1) + M_k' L_k' + M_k'' L_k'' - M_k L_k \\ &\leq S(f, D_1) \end{aligned}$$

because

$$M_k' L_k' + M_k'' L_k'' \leq M_k (L_k' + L_k'') = M_k L_k.$$

The rest is an exercise. □

Theorem (Every upper sum exceeds every lower sum)

Let D_1 and D_2 be any two dissections of $[a, b]$. Then

$$S(f, D_1) \geq s(f, D_2).$$

Proof.

Let D_3 be the new dissection formed by taking all the distinct points of D_1 and D_2 in increasing order: D_3 is a refinement of both D_1 and D_2. Hence

$$S(f, D_3) \leq S(f, D_1) \quad \text{and} \quad s(f, D_3) \geq s(f, D_2).$$

But

$$S(f, D_3) \geq s(f, D_3),$$

so

$$S(f, D_1) \geq s(f, D_2).$$

□

Theorem

$$\overline{\int_a^b} f = \inf_D f \geq \underline{\int_a^b} f = \sup_D f.$$

Proof.

For any dissection D_2 we have

$$\overline{\int_a^b} f = \inf_{D_1} S(f, D_1) \geq s(f, D_2).$$

Hence

$$\overline{\int_a^b} f \geq \sup_{D_2} s(f, D_2) = \underline{\int_a^b} f.$$

□

Definition (Riemann integrable)

Let $f : [a, b] \to \mathbb{R}$*. If*

$$\overline{\int_a^b} f = \inf_D f = \underline{\int_a^b} f = \sup_D f,$$

then we say that f *is* **Riemann integrable**.

Example (Non-Riemann integrable function)

Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q}. \end{cases}$$

For any dissection D of $[a, b]$ we have $S(f, D) = b - a$ and $s(f, D) = 0$. Hence

$$\overline{\int_a^b} f = \inf_D f = b - a > 0 = \underline{\int_a^b} f = \sup_D f$$

and f is **not** Riemann integrable.

The need to integrate functions like f led to the Lebesgue integral in the early 20th century, but it's best to start with Riemann integration first.

Theorem (Increasing functions are Riemann integrable)

If $f : [0,1] \to \mathbb{R}$ is an increasing function, then it's Riemann integrable.

Proof.

Let $x_k = k/n$, for $k = 0, 1, \ldots, n$ be our dissection D_n. Then

$$S(f, D_n) = \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n})$$

while $s(f, D_n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\frac{k}{n}) = S(f, D_n) + \frac{1}{n}\Big(f(0) - f(1)\Big)$.
Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ for which

$$S(f, D_n) - \epsilon \le s(f, D_n) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le S(f, D_n).$$

□

Theorem (Continuous functions are Riemann integrable)

Every continuous $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof.

Crucial fact: f is uniformly continous: given any $\epsilon > 0$, there exists $\delta > 0$ for which $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, for any $x, y \in [a, b]$.

If D is any dissection of $[a, b]$ for which $L < \delta$, then $M_k - m_k < \epsilon$ and

$$S(f, D) - s(f, D) < \epsilon(b - a).$$

□

Example (Nonexaminable: $\mathbb{Q}$ has measure zero)

How do we measure the length of a set like $\mathbb{Q} \cap (0,1)$? We saw this in Lecture 1: Let $q_1, q_2, \ldots$ denote the rational numbers in $(0,1)$. Given any $\epsilon > 0$, let

$$I_n = (q_n - \frac{\epsilon}{2^{n+1}}, q_n + \frac{\epsilon}{2^{n+1}}), \qquad \text{for } n \in \mathbb{N}.$$

Then $q_n \in I_n$ and the length of I_n is $L_n = \frac{\epsilon}{2^n}$. Thus $\mathbb{Q} \cap (0,1)$ is contained in $I_1 \cup I_2 \cup \cdots$ and the "length" of $\mathbb{Q} \cap (0,1)$ should be less than

$$L_1 + L_2 + L_3 + \cdots = \epsilon \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) = \epsilon.$$

Since $\epsilon > 0$ can be as small as we wish, we say that $\mathbb{Q} \cap (0,1)$ has **measure zero**, while $(0,1)$ has measure 1.

Definition (Nonexaminable: Outer measure and measure zero)

Given any subset A of $(0,1)$, we say that an **open cover** *is any sequence of open intervals $I_k = (a_k, b_k)$ in $(0,1)$ for which*

$$A \subset \cup_{k=1}^{\infty} I_k.$$

We define the **outer measure** *of A to be*

$$\mu^*(A) = \inf \sum_{k=1}^{\infty} \ell(I_k)$$

where the $\inf$ is over all open covers of A and $\ell(I_k) = b_k - a_k$. We say that A has **measure zero** *if $\mu^*(A) = 0$.*

Example (Nonexaminable: Countable sets have measure zero)

Let $x_1, x_2, x_3, \ldots$ be any real sequence and choose any $\epsilon > 0$. Define

$$I_k = (x_k - \frac{\epsilon}{2^{k+1}}, x_k + \frac{\epsilon}{2^{k+1}}).$$

Then (I_k) is an open cover of the sequence and

$$\ell(I_k) = \frac{\epsilon}{2^k}.$$

Hence

$$\sum_{k=1}^{\infty} \ell(I_k) = \epsilon$$

and $\mu^*(\{x_1, x_2, \ldots\}) = 0$.

Theorem (Nonexaminable)

A function $f : (a, b) \to \mathbb{R}$ is Riemann integrable if and only if its set of discontinuities has measure zero: we say that f is **almost everywhere continuous**.

Theorem (Nonexaminable)

An increasing function is almost everywhere continuous.

Proof.

If an increasing function $f : \mathbb{R} \to \mathbb{R}$ is discontinuous at a point $a \in \mathbb{R}$, then that corresponds to an open interval (a jump) in $f(\mathbb{R})$. The discontinuities of f therefore correspond to disjoint open intervals of $f(\mathbb{R})$. Every open interval contains a rational number, so there are countably many disjoint open intervals. □