Real Analysis 8: Integration and Taylor Series

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You can download these slides and the lecture videos from my office server

`http://econ109.econ.bbk.ac.uk/brad/analysis/`

Recommended books: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.
J. C. Burkill (1978), "A First Course in Mathematical Analysis", Cambridge University Press.

Example

Let

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & x \in \mathbb{R} \setminus \{0\}. \end{cases}$$

Then f is integrable on $[-A, A]$, for any $A > 0$, and

$$\int f(x)\, dx = 0.$$

The idea is to choose the dissection

$$D_n = \{-A, -\frac{1}{n}, 0, \frac{1}{n}, A\}.$$

Then $S(f, D_n) = 2/n$ and $s(f, D_n) = 0$.

Theorem

If $f : [a, b] \to [0, \infty)$ is continuous and $f(c) > 0$ for some $c \in [a, b]$, then

$$\int_a^b f(x)\, dx > 0.$$

Proof.

There exists $\delta > 0$ for which $f(x) > f(c)/2$ for

$$x \in I = (c - \delta, c + \delta) \cap [a, b].$$

Thus

$$\int_I f(x)\, dx > \frac{f(c)}{2} \int_I dx > 0$$

and

$$\int_a^b f(x)\, dx \geq \int_I f(x)\, dx$$

because $f(x) \geq 0$ for all $x \in [a, b]$. □

Theorem

If $f : [a, b] \to [0, \infty)$ *is continuous and*

$$\int_a^b f(x)\, dx = 0$$

then $f(x) = 0$ *for all* $x \in [a, b]$.

Theorem (Fundamental Theorem of Calculus)

Let $f\colon (a,b) \to \mathbb{R}$ *be continuous and choose any* $u \in (a,b)$*. Then* $F\colon (a,b) \to \mathbb{R}$ *defined by the integral*

$$F(t) = \int_u^t f(x)\,dx, \qquad t \in (a,b),$$

is differentiable in (a,b) *and* $F'(t) = f(t)$ *for all* $t \in (a,b)$*.*

Proof.

$$\begin{aligned} |F(t+h) - F(t) - hf(t)| &= \left| \int_t^{t+h} f(x) - f(t)\,dx \right| \\ &\le h \sup_{x \in [t,t+h]} |f(x) - f(t)|. \end{aligned}$$

Given any $\epsilon > 0$, there exists $h > 0$ for which $|f(x) - f(t)| < \epsilon$ for $|x - t| \le h$.

□

Theorem

Suppose $f : (a, b) \to \mathbb{R}$ *is continuous,* $u \in (a, b)$ *and* $c \in \mathbb{R}$*. Then there exists a unique solution to the differential equation*

$$g'(t) = f(t), \qquad \text{for } t \in (a, b),$$

such that $g(u) = c$*.*

Proof.

We know that

$$g(t) = c + \int_u^t f(x)\,dx$$

is a solution. However, if g_1 and g_2 are any two solutions, then $g_1'(t) - g_2'(t) = 0$, so $g_2(t) - g_1(t)$ is constant and, since $g_1(u) = g_2(u) = c$, we deduce that this constant is zero. □

Theorem

Suppose $g\colon (\alpha, \beta) \to \mathbb{R}$ *is continuous and differentiable and* $[a, b]$ *is contained in the open interval* (α, β)*. Then*

$$\int_a^b g'(x)\,dx = g(b) - g(a).$$

Proof.

Define

$$U(t) = \int_a^t g'(x)\,dx - g(t) + g(a),$$

for $\alpha < t < \beta$. Then

$$U'(t) = g'(t) - g'(t) = 0,$$

i.e. $U(t)$ is constant. But $U(a) = 0$, so this constant must be zero. □

Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $g : [\gamma, \delta] \to \mathbb{R}$ is continuously differentiable (i.e. g' is continuous). Suppose further that $g([\gamma, \delta]) \subset [a, b]$. Then, for $c, d \in [\gamma, \delta]$,

$$\int_{g(c)}^{g(d)} f(s)\, ds = \int_c^d f(g(x))g'(x)\, dx.$$

Proof.

Define

$$F(t) = \int_a^t f(u)\, du.$$

Then

$$\frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Hence

$$\int_{g(c)}^{g(d)} f(s)\, ds = F(g(d)) - F(g(c))$$

Theorem

Suppose that $f : [a, b] \to \mathbb{R}$ *is continuously differentiable and* $g : [a, b] \to \mathbb{R}$ *is continous. If* $G : [a, b] \to \mathbb{R}$ *is any* **primitive** *of* g*, i.e.* $G'(x) = g(x)$*, then*

$$\int_{-a}^{b} f(x)g(x)\,dx = [f(x)G(x)]_a^b - \int_a^b f'(x)G(x)\,dx.$$

Proof.

$$\begin{aligned}
&\int_a^b f(x)g(x)\,dx + \int_a^b f'(x)G(x)\,dx \\
&= \int_a^b f(x)g(x) + f'(x)G(x)\,dx \\
&= \int_a^b \frac{d}{dx}\left(f(x)G(x)\right)\,dx \\
&= [f(x)G(x)]_a^b.
\end{aligned}$$

Theorem (Taylor 1)

$$I_1(x) = \int_a^x f'(t)\,dt = f(x) - f(a).$$

Theorem (Taylor 2)

If

$$I_2(x) = \int_a^x (x-t)\,f^{(2)}(t)\,dt,$$

then

$$I_2(x) = f(x) - f(a) - f'(a)\,(x-a)\,.$$

Proof.

$$\begin{aligned} I_2(x) &= \Big[(x-t)\,f'(t)\Big]_{t=a}^{t=x} + \int_a^x f'(t)\,dt \\ &= -f'(a)\,(x-a) + f(x) - f(a) \\ &= f(x) - f(a) - f'(a)\,(x-a)\,. \end{aligned}$$

Theorem (Taylor 3)

Define

$$I_n(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt, \qquad n \ge 1.$$

Then

$$I_n(x) = -\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + I_{n-1}(x), \qquad \text{for } n \ge 2.$$

Proof.

Integrate by parts:

$$I_n(x)$$

$$= \left[\frac{(x-t)^{n-1}}{(n-1)!} f^{(n-1)}(t)\right]_{t=a}^{t=x} - \int_a^x \left[\frac{-(x-t)^{n-2}}{(n-2)!}\right] f^{(n-1)}(t)\,dt$$

$$= -\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + I_{n-1}(x), \qquad \text{for } n \ge 2.$$

Theorem (Taylor 4)

$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + I_n(x).$$

Proof.

$$\begin{aligned}
I_n(x) &= -\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + I_{n-1}(x) \\
&= -\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{(x-a)^{n-2}}{(n-2)!} f^{(n-2)}(a) + I_{n-2}(x) \\
&= \cdots \\
&= -\sum_{k=1}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + I_1(x) \\
&= f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a).
\end{aligned}$$

Example

Let $f(x) = (1+x)^{-1}$, for $x \in (-1,1)$. Then

$$f^{(n)}(x) = (-1)^n n!(1+x)^{-n-1}.$$

Thus $f^{(n)}(0) = (-1)^n n!$ and

$$p_n(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n-1} (-1)^k x^k.$$

The limit is then

$$\frac{1}{1+x} = \sum_{k=0}^{n-1} (-1)^k x^k,$$

for $|x| < 1$.

Example (Binomial theorem)

Let

$$f(x) = (1+x)^n,$$

for $x \in \mathbb{R}$, where n is a positive integer. Then

$$f^{(k)}(x) = n(n-1)(n-2)\cdots(n-k+1)(1+x)^{n-k},$$

for $k \le n$, but $f^{(k)}(x) \equiv 0$, for $k > n$. Hence Taylor's theorem gives

$$(1+x)^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} x^k.$$

Example

Let $f(x) = (1+x)^{-1/2}$, for $x \in (-1,1)$. Then

$$
\begin{aligned}
f'(x) &= (-1/2)(1+x)^{-3/2},\\
f^{(2)}(x) &= (-1/2)(-3/2)(1+x)^{-3/2},\\
f^{(3)}(x) &= (-1/2)(-3/2)(-5/2)(1+x)^{-7/2},\\
f^{(4)}(x) &= (-1/2)(-3/2)(-5/2)(-7/2)(1+x)^{-9/2},\\
f^{(k)}(x) &= \frac{(-1)^k(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1(1+x)^{-(2k+1)/2}}{2^k}.
\end{aligned}
$$

Thus the Taylor series is

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty}\left(\frac{(-1)^k(2k-1)(2k-3)(2k-5)\cdots 5\cdot 3\cdot 1}{2^k k!}\right)x^k$$

Example

Further

$$\begin{aligned}
&\frac{(-1)^k(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}{2^k k!} \\
&= \frac{(-1)^k(2k)!}{2^k k!(2k)(2k-2)(2k-4)\cdots 4\cdot 2} \\
&= \frac{(-1)^k(2k)!}{4^k\,(k!)^2} \\
&= (-1)^k\binom{2k}{k}4^{-k},
\end{aligned}$$

whence

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}4^{-k}x^k,$$

for $|x| < 1$.

Exercise

Show that

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$\text{and} \qquad (1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \cdots .$$

Example

$$
\begin{aligned}
(n+1)^{1/2} &= n^{1/2}\left(1+\frac{1}{n}\right)^{1/2} \\
&= n^{1/2}\left(1+\frac{1}{2n}-\frac{1}{8n^2}+\cdots\right) \\
&= n^{1/2}+\frac{1}{2n^{1/2}}-\frac{1}{8n^{3/2}}+\cdots,
\end{aligned}
$$

which implies that

$$
a_n = \sqrt{n+1}-\sqrt{n} = \frac{1}{2n^{1/2}}-\frac{1}{8n^{3/2}}+\cdots,
$$

and thus $a_n \to 0$ as $n \to \infty$.

Exercise

Use Taylor series to study the convergence, or otherwise, of

$$b_n = (n+1)^{1/3} - n^{1/3},$$

as $n \to \infty$*.*

Definition (Infinite Integrals)

If $f : [a, \infty) \to \mathbb{R}$ *is integrable on* $[a, b]$ *for every* $b > a$ *and*

$$\int_a^b f(x)\,dx \to L$$

as $b \to \infty$, *then we say that*

$$\int_a^\infty f(x)\,dx$$

exists with value L.

Theorem

If $f : [a, \infty) \to [0, \infty)$ is integrable on $[a, b]$ for every $b > a$, then

$$\int_a^\infty f(x)\,dx$$

exists if and only if there exists K for which

$$\int_a^b f(x)\,dx \le K \qquad \text{for all } b > a.$$

Proof:

If the infinite integral exists, then

$$u_n = \int_a^n f(x)\,dx \qquad \text{for } n \ge a.$$

is an increasing convergent sequence of positive numbers. Hence it's bounded: there exists K for which $u_n \le K$ for all n. Given any $b > a$, we therefore have $\int_a^b f(x)\,dx \le u_N \le K$ for $N > b$.

Conversely, suppose (u_n) is convergent with limit L. For any $b > a$, there exists $N \in \mathbb{N}$ for which

$$u_N \le \int_a^b f(x)\,dx \le u_{N+1}.$$

Hence

$$\int_a^b f(x)\,dx \to L$$

as $b \to \infty$. $\square$