Real Analysis 4-5: The Integral Test for Series

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You can download these slides and the lecture videos from my office server

http://econ109.econ.bbk.ac.uk/brad/analysis/

Recommended books: Lara Alcock (2014), "How to Think about Analysis", Oxford University Press.
J. C. Burkill (1978), "A First Course in Mathematical Analysis", Cambridge University Press.

Riemann integration is studied in depth in Lecture 7.
BUT there is a simple Integral Test that relates series to integral of functions for which you only need to know that decreasing and increasing functions are Riemann integrable and have the following simple property:

Theorem

If $f_1(x) \ge f_2(x)$ for $x \in [a, b]$, then

$$\int_a^b f_1(x)\,dx \ge \int_a^b f_2(x)\,dx.$$

This lecture will enable you to complete Q7 on HW3.

Theorem (Integral Test for series convergence)

Let $f : [1, \infty) \to (0, \infty)$ *be a* **positive decreasing** *function. Then*

1. $$\sum_{n=1}^{\infty} f(n) \quad \textit{and} \quad \int_1^{\infty} f(x)\, dx$$

   either both diverge or both converge.

2. *Let*

   $$a_n = \sum_{k=1}^{n} f(k) - \int_1^{n} f(x)\, dx, \qquad n \in \mathbb{N}.$$

   Then (a_n) *is a decreasing sequence and* $0 \le a_n \le f(1)$, *so it's convergent with limit in* $[0, f(1)]$.

Proof.

1. If $k-1 \le x \le k$, then $f(k-1) \ge f(x) \ge f(k)$. Integrating over $(k-1, k)$,

$$f(k-1) \ge \int_{k-1}^{k} f(x)\,dx \ge f(k).$$

Adding these inequalities

$$\sum_{k=1}^{n-1} f(k) \ge \int_{1}^{n} f(x)\,dx \ge \sum_{k=2}^{n} f(k). \qquad (*)$$

Hence $\int_1^\infty f$ and $\sum_1^\infty f(n)$ both converge or both diverge.

2. We have

$$a_n - a_{n-1} = f(n) - \int_{n-1}^{n} f(x)\,dx \le 0$$

and $0 \le a_n \le f(1)$ by (*) (exercise).

Example ($\sum n^{-3/2}$ convergent)

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is convergent if and only if the infinite integral

$$\int_1^{\infty} x^{-3/2}\, dx$$

exists. But

$$\int_1^{A} x^{-3/2}\, dx = \left[-2x^{-1/2}\right]_1^{A} = 2 - \frac{2}{A^{1/2}} \to 2,$$

as $A \to \infty$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is convergent.

Example (Harmonic series $\sum 1/n$ is divergent)

Let $f(x) = 1/x$. Here's the Integral Test in detail (draw a picture!):

$$S_N = \sum_{n=1}^{N} \frac{1}{n} \geq \int_1^{N+1} \frac{1}{x}\, dx = \ln(N+1).$$

Since $\ln(N+1) \to \infty$ as $N \to \infty$, the partial sums (S_N) are unbounded. Hence the harmonic series is divergent.

Logarithms: Here we are using

$$\ln y = \int_1^y \frac{1}{x}\, dx, \qquad \text{for } y > 0,$$

which we shall discuss in Lecture 7.

Example ($\sum 1/n^c$ is divergent for $0 < c < 1$)

Here's the Integral Test in detail again (draw a picture!):

$$S_N = \sum_{k=1}^{N} \frac{1}{n^c} \geq \int_1^{N+1} x^{-c}\, dx = \left[\frac{x^{1-c}}{1-c}\right]_1^{N+1} = \frac{(N+1)^{1-c} - 1}{1-c}.$$

But, if $c < 1$, then $1 - c > 0$ and $(N+1)^{1-c} \to \infty$ as $N \to \infty$. Thus the partial sums (S_N) are unbounded and the series is divergent.

Example ($\sum 1/n^c$ is convergent for $c > 1$)

Here's the Integral Test in detail again (draw a picture!):

$$\sum_{k=2}^{N} \frac{1}{n^c} \le \int_1^N x^{-c}\,dx = \left[\frac{x^{1-c}}{1-c}\right]_1^N = \frac{N^{1-c}-1}{1-c}.$$

But, if $c > 1$, then $1 - c < 0$ and $N^{1-c} \to 0$ as $N \to \infty$. Thus the partial sums (S_N) are bounded and the series is convergent.

Theorem (Euler's constant)

Let

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n.$$

Then $a_n \to \gamma$, *where* $0 < \gamma < 1$.

Proof.

Apply the Integral Test to $f(x) = 1/x$. $\square$

Example (Integrals and Sequences)

Let $r > 0$ and define the sequence

$$a_n = \left(1 + \frac{r}{n}\right)^n, \qquad n \in \mathbb{N}.$$

You should know that $a_n \to \exp r$, but here we prove that it's a strictly increasing sequence using the integral

$$\ln a_n = n \int_1^{1+r/n} \frac{1}{s}\, ds = \int_0^r \frac{1}{1 + u/n}\, du,$$

using the substitution $s = 1 + u/n$. Then the inequality $u/n > u/(n+1)$, for $u > 0$, implies that

$$\frac{1}{1 + u/n} < \frac{1}{1 + u/(n+1)}, \qquad \text{for } u > 0.$$

Hence $\log a_n < \log a_{n+1}$.